Tractable Bayesian Estimation of Non-Linear DSGE Models Using Higher-Order Approximations--with Application to a Model with Global Banking

Robert Kollmann (*)
ECARES, Université Libre de Bruxelles & CEPR
June 14, 2015

This paper discusses a tractable novel method for Bayesian estimation of non-linear Dynamic Stochastic General Equilibrium (DSGE) models that are solved using second- and third order accurate approximations. I apply the Kalman filter to a state-space representation of second- and third-order accurate model solutions based on the 'pruning' scheme of Kim et al. (2008). The method exploits the fact that the 'pruned' system is *linear* in a state vector that consists of predetermined and exogenous variables, and of products of those variables. By contrast to particle filters, no stochastic simulations are needed for the method here. The method here is, hence, much faster and it is thus suitable for the estimation of medium-scale models. Monte Carlo experiments suggest that the method here provides reliable estimates of model parameters and of latent state variables, even for highly non-linear economies with big shocks. The estimation routines developed here are based on decision rules computed using the Dynare package.

The paper also uses the method to empirically estimate a two-country RBC model with global banks and financial shocks (Kollmann, 2013). Compared to the linearized model, the non-linear model yields markedly higher estimates of the unconditional variance of real activity accounted for by banking shocks; the estimated non-linear model also suggests that banking shocks account for a much higher share of the drop in output, investment and employment during the global financial crisis.

- -

(*) I thank Matteo Iacoviello, Johannes Pfeifer, Kostas Theodoridis and workshop participants at the Sheffield Macroeconomics conference for useful discussions. Thank are also due to Johannes Pfeifer for advice about the Dynare software. Financial support from Académie Universitaire Wallonie-Bruxelles (Action de recherche concertée, grant ARC-AUWB/2010-15/ULB-11) and from the European Commission (FP7 program; Macfinrobods, grant 612796) is gratefully acknowledged.
Address: ECARES, CP 114, ULB; 50 Av. F. Roosevelt, B-1050 Brussels, Belgium
robert_kollmann@yahoo.com

Tractable Bayesian Estimation of Non-Linear DSGE Models Using Higher-Order Approximations--with Application to a Model with Global Banking

Robert Kollmann (ECARES, Université Libre de Bruxelles & CEPR)

June 14, 2015

Since global financial crisis: strong interest in macro models with financial frictions/shocks and in nonlinearity

- **Financial health is important state variables**
- **Leverage of financial system, household/firm/government debt affects response to shocks: when leverage is high, then effect of adverse shocks magnified.**
- **Fiscal policy might have stronger effect on GDP in slump, especially at ZLB: fiscal multipliers depend on state of economy.**
- **Asymmetry between positive & negative shocks**
- **Asymmetric price/wage adjustment costs (downward rigidity)**

Much recent work develops/estimates linearized medium-scale DSGE models with financial sector.

Examples of ESTIMATED LINEARIZED models with banks: eg, Gerali et al. (2010); Kollmann, Ratto & Roeger (2013); Kollmann (2013). Find that financial shocks have modest role in 'normal' times, matter more in crises. LINEARIZATION may miss important effects.

Theoretical literature on financial frictions often uses highly stylized models with exact non-linear solutions

E.g. Brunnermeier & Sannikov (2014) (many others).

- **Argue that rare financial crises affect behavior in 'normal times'**
- **These models too simple for reliable <u>empirical</u> analysis. Realistic analysis requires medium-scale models with many shocks & state variables**

● **Challenge: construction medium-scale non-linear DSGE models that can be taken to data in tractable manner.**

Only tractable solution method for *medium-scale* DSGE models:

2nd (or higher) order Taylor series approximations of policy functions (around deterministic steady state).

● **Approximations of order 2 or 3 can be computed very easily and FAST . Jin & Judd (2000), Sims (2000), Collard & Juillard (2002), Schmitt-Grohé & Uribe (2004), Kollmann (2004), Lombardo & Sutherland (2007), Kim, Kollmann & Kim (2010), Kollmann, Kim & Kim (2011). Very widely used in macroeconomics.**

Especially user-friendly code: Dynare (Adjemian et al., 2011)

● Challenge: how to take 2nd/3rd order approximated models to data?

●Possible estimation approach: simulated method of moments (SMM)

E.g. Andreasen, Fernandez-Villaverde & Rubio-Ramirez (2014)

Pick model parameters such that selected predicted model moments are closest to empirical moments.

Drawbacks:

► SMM results can be sensitive to selected moments

► SMM does not generate estimates of latent states; thus cannot estimate historical decompositions (contributions of different shocks to data)

● This paper: Likelihood-based = maximize predictive ability of model

In line with standard likelihood-based empirical estimates of ***linearized*** models (eg Kim (1999), Otrok (2000), Smets & Wouters (2007))

Likelihood computation (prediction error decomposition): requires filtered estimates of states. How to achieve this for non-linear model?

KEY INGREDIENT OF APPROACH HERE: PRUNING

● When simulating higher-order approximated model: common to use 'pruning' scheme under which second-order terms are replaced by products of linearized solution etc.

Pruning of n-th order approx. model implies that endogenous variables depend on powers of exogenous innovations $(\varepsilon_t)^k$ k=1,..,n.

● Unless pruning is used, higher-order approximated models often generate exploding simulated time paths

$\Rightarrow$ pruning crucial for applied work based on higher-order approx.

● This paper assumes that PRUNED 2nd (or 3rd) order approximated model is TRUE data generating process (DGP)

● Method here exploits fact that PRUNED n-th order approximated model is LINEAR in a state vector consisting of variables solved to orders 1,..,n, and in products of variables solved to orders 1,..,n-1.

● Allows convenient closed-form determination of conditional mean and variance of state vector

● Key idea of this paper: apply linear updating rule of standard Kalman filter to pruned state equation

● Method here is MUCH faster than particle filters, as it is not based on stochastic simulations.

Monte Carlo experiments show: deterministic filter here is more accurate than standard particle filter, especially with big shocks & high curvature

● High speed of filter allows to estimate structural model parameters

● Kollmann (Computational Economics, 2015) derives filter for models solved to 2nd order using gensys2 method (Sims, 2000).

● This paper shows how to derive filter for models solved to 2nd order and 3rd order, using Dynare. Dynare allows great gain in speed.

● Remainder of talk:

► Key ideas of method

► Empirical application to DGSE model with banking sector (Kollmann, JMCB 2013).

Key finding: Non-linearities matter empirically.

1) In estimated non-linear model, financial shocks are more important than in estimated linearized model.

2) Responses of macro variables to exogenous shocks are state-contingent

KEY IDEAS OF METHOD

Generic DSGE model can be written as:

$$E_t M(S_{t+1}, Y_{t+1}, S_t, Y_t, \varepsilon_{t+1}) = 0,$$

S_{t+1}: date t+1 predetermined variables (set at t) & exogenous variables (realized at t)

Y_{t+1}: jump variables (co-states) at t+1

ε_{t+1}: exogenous i.i.d. innovations; $E_t \varepsilon_{t+1} = 0; Var_t \varepsilon_{t+1} = \xi^2 \cdot \mathrm{I}$; ξ: scalar (shock size)

Model solution given by decision rule:

$$S_{t+1} = F(S_t, \varepsilon_{t+1}, \xi), \quad Y_{t+1} = G(S_t, \varepsilon_{t+1}, \xi)$$

Deterministic steady state (SS): $S = F(S,0,0); \; Y = G(S,0,0).$

Compute n-th order Taylor series expansion of decision rule around SS.

Let $s_t \equiv S_t - S; \; y_t \equiv Y_t - Y.$

Approximate model solutions

- **First-order:**

$$s_{t+1} = F_1 s_t + F_2 \varepsilon_{t+1} \qquad (1)$$

- **Second-order (state contingent effects of innovations)**

$$s_{t+1} = F_0 \xi^2 + F_1 s_t + F_2 \varepsilon_t + F_{11} s_t \otimes s_t + F_{12} s_t \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \qquad (2)$$

- **Third-order (state contingent conditional variance, risk premia)**

$$s_{t+1} = F_0 \xi^2 + (F_1 + F_{1\xi} \xi^2) s_t + (F_2 + F_{2\xi} \xi^2) \varepsilon_{t+1} + F_{11} s_t \otimes s_t + F_{12} s_t \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} + \ldots$$

$$F_{111} s_t \otimes s_t \otimes s_t + F_{112} s_t \otimes s_t \otimes \varepsilon_{t+1} + F_{122} s_t \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} + F_{222} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} \qquad (3)$$

ISSUE: In repeated applications of (2), (3) higher -order terms of state variables appear e.g., when s_{t+1} is quadratic in s_t, then s_{t+2} is quartic in s_t.

Pruning removes these terms of increasing order.

Motivation for pruning: (2) & (3) have spurious steady states (not present in the original model)--some of these steady states mark transitions to unstable behavior. Large shocks can move the model into an unstable region. Pruning overcomes this problem. If the first-order solution is stable, then the pruned higher-order solutions too are stationary

$a_t^{(n)}$: **variable solved to n-th order accuracy.**

$a_t = a_t^{(1)} + R^{(2)}$**, with** $R^{(2)}$**: terms of order 'n' or higher in deviations from SS.**

$$(a_t b_t) = (a_t^{(1)} + R^{(2)})(b_t^{(1)} + R^{(2)}) = a_t^{(1)} b_t^{(1)} + R^{(3)} \quad \Rightarrow \quad (a_t b_t)^{(2)} = a_t^{(1)} b_t^{(1)}.$$

Similarly: $(a_t b_t c_t)^{(3)} = a_t^{(1)} b_t^{(1)} c_t^{(1)}$ **and** $(a_t b_t)^{(2)} = a_t^{(2)} b_t^{(1)} + a_t^{(1)} (b_t^{(2)} - b_t^{(1)})$

PRUNING SCHEME (Kim el al. (2008)):

► **In second-order accurate model solution, replace** $s_t \otimes s_t$ **by** $s_t^{(1)} \otimes s_t^{(1)}$

► **In third-order accurate model solution, replace** $s_t \otimes s_t$ **by** $s_t^{(2)} \otimes s_t^{(1)} + s_t^{(1)} \otimes (s_t^{(2)} - s_t^{(1)})$

and replace $s_t \otimes s_t \otimes s_t$ **by** $s_t^{(1)} \otimes s_t^{(1)} \otimes s_t^{(1)}$

$$s_{t+1}^{(1)} = F_1 s_t^{(1)} + F_2 \varepsilon_t$$

$$s_{t+1}^{(2)} = F_0 \xi^2 + F_1 s_t^{(2)} + F_2 \varepsilon_t + F_{11} s_t^{(1)} \otimes s_t^{(1)} + F_{12} s_t^{(1)} \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1}$$

If $\{s_{t+1}^{(1)}\}$ **is stationary, then** $\{s_{t+1}^{(2)}\}$ **too is stationary.**

For $n\,x1$ **vector** x, **let** $P_2(x) = [x_1x_1, x_1x_2, .., x_1x_n, x_2x_2, .., x_2x_n, .., x_{n-1}x_{n-1}, x_{n-1}x_n, x_nx_n]$.

be vector of all $\frac{1}{2}n(n+1)$ **(cross-)products of elements of** x.

Can write $F_{11}s_t^{(1)} \otimes s_t^{(1)} = F_{11}P_2(s_t^{(1)})$

$$P_2(s_{t+1}^{(1)}) = K_{11}P_2(s_{t+1}^{(1)}) + K_{12}s_t^{(1)} \otimes \varepsilon_{t+1}^{(1)} + K_{22}P_2(\varepsilon_{t+1})$$

KEY INSIGHT: Pruned system can be written as a LINEAR system in terms of levels and products of STATE variables:

$$Z_{t+1} = H_0 + H_1 Z_t + u_{t+1}, \quad E_t u_{t+1} = 0.$$

For second-order accurate model:

$$Z_{t+1} \equiv [s_{t+1}^{(2)}; s_{t+1}^{(1)}; P_2(s_{t+1}^{(1)})];\ u_{t+1} \equiv H_2 \varepsilon_{t+1} + H_{12}(s_t^{(1)} \otimes \varepsilon_{t+1}) + H_{22}(P(\varepsilon_{t+1}) - EP(\varepsilon_{t+1}))$$

Third-order system:

$$Z_{t+1} \equiv [s_{t+1}^{(3)}; s_{t+1}^{(2)}; s_{t+1}^{(1)}; s_{t+1}^{(2)} \otimes s_{t+1}^{(1)}; P_2(s_{t+1}^{(1)}); P_3(s_{t+1}^{(1)})];$$

$$u_{t+1} \equiv H_2 \varepsilon_{t+1} + H_{12}(s_t^{(1)} \otimes \varepsilon_{t+1}) + H_{12}(s_t^{(2)} \otimes \varepsilon_{t+1}) + H_{22}(P_2(\varepsilon_{t+1}) - EP_2(\varepsilon_{t+1})) +$$
$$H_{112} P_2(s_t^{(1)}) \otimes \varepsilon_{t+1} + H_{122} s_t^{(1)}(P_2(\varepsilon_{t+1}) - EP_2(\varepsilon_{t+1})) + H_{222} P_3(\varepsilon_{t+1})$$

Straightforward (but tedious) to compute moments of state vector.

CO-STATES:

Co-states can be expressed as function of states: $Y_{t+1} = J(S_{t+1})$

2nd/3rd order approximation: $y_{t+1} \equiv Y_{t+1} - Y = K \cdot Z_{t+1}$

OBSERVATION EQUATION:

At t=1,..,T analyst observes vector x_t^{obs} **: linear function of (co-)states**

$$x_t^{obs} = \Xi \cdot Z_t + \lambda_t$$

λ_t **: i.i.d. measurement error.**

THE FILTER

Apply linear updating equation of standard Kalman filter to system

$$Z_{t+1} = H_0 + H_1 Z_t + u_{t+1},\ \ x_t^{obs} = \Xi \cdot Z_t + \lambda_t.$$

$$\Rightarrow$$

$$E_t Z_{t+1} = H_0 + H_1 E_t Z_t,$$

$$V_t Z_{t+1} = H_1 V_t Z_t H_1' + V_t(u_{t+1})$$

$$E_{t+1} Z_{t+1} = E_t Z_{t+1} + \phi_t \cdot (x_{t+1}^{obs} - E_t x_{t+1}^{obs}),$$ **with** $E_t x_{t+1}^{obs} = \Xi \cdot E_t Z_{t+1}$

and $\phi_t \equiv V_t(Z_{t+1})\Xi'\{\Xi V_t(Z_{t+1})\Xi' + V(\lambda_{t+1})\}^{-1}$

$$V_{t+1}(Z_{t+1}) = V_t(Z_{t+1}) - V_t(Z_{t+1})\Xi'\{\Xi V_t(Z_{t+1})\Xi' + V(\lambda_{t+1})\}^{-1}\Xi V_t(Z_{t+1})$$

Key issue for implementation: high dimension of augmented state vector (Z).

	dim(Z) for 2nd ord. approx.	dim(Z) for 3rd ord. approx.
n=5 states	25	90
n=10 states	75	405
n=20	250	2210

Reduce dimension of linear system using eigen-decomposition.

$$Z_{t+1} = H_0 + H_1 Z_t + u_{t+1},$$

$$r \equiv rank(V(Z)) < m \equiv \dim(Z)$$

Λ_1**: r** x **r matrix with of r positive eigenvalues of** $V(Z)$ **on main diagonal**

W_1**: m** x **r matrix of associated eigenvectors of** $V(Z)$ **with** $W_1 ' W_1 = I_r$

$$\Rightarrow V(Z)W_1 = W_1 \Lambda_1 \quad \Rightarrow W_1 ' V(Z) W_1 = \Lambda_1$$

Then can write $Z_{t+1} - E(Z_{t+1}) = W_1 z_{t+1}$ **for r** x **1 vector** z_{t+1} **s.t.** $E(z_{t+1}) = 0$ **&** $V(z_{t+1}) = \Lambda_1$

Write system in terms of lower-dimensional state vector z_{t+1} :

$$Z_{t+1} = H_0 + H_1 Z_t + u_{t+1} \ \ \& \ \ Z_{t+1} - E(Z_{t+1}) = W_1 z_{t+1}$$

$$\Rightarrow z_{t+1} = W_1 ' H_1 W_1 \cdot z_t + W_1 ' u_{t+1}$$

Observation equation:

$$x_t^{obs} = \Xi \cdot Z_t + \lambda_t \quad \Rightarrow x_t^{obs} = \Xi \cdot E(Z) + \Xi \cdot W_1 \cdot z_t + \lambda_t$$

Apply Kalman filter to lower-dimensional system

$\Rightarrow$ for given values of structural model parameters can generate filtered and smoothed estimates of the state vectors z_t & Z_t.

BAYESIAN ESTIMATION OF MODEL PARAMETERS

Use prior distribution of structural model parameters $p(\theta)$; maximize posterior log-likelihood function based on multivariate normal distribution.

NB Disturbance in pruned state equation is non-Gaussian; however maximization of Gaussian likelihood produces consistent and asymptotically normal parameter estimates; e.g., Hamilton (ch .13, 1994).

Filter accuracy:

Kalman filter applied to pruned state-space is more accurate than standard particle filter, and MUCH faster.

With big shocks and strong curvature, the increase in accuracy is substantial (RMSEs can be orders of magnitude smaller).

High speed makes parameter estimation feasible. Parameters tightly estimated.

Illustration: Monte Carlo for basic RBC model

$$V_t = \{\tfrac{1}{1-\sigma} C_t^{1-\sigma} - \tfrac{1}{1+1/\eta} N_t^{1+1/\eta}\} + \lambda_t \beta E_t V_{t+1}, \quad C_t + I_t = Y_t, \quad Y_t = \theta_t K_t^{\alpha} N_t^{1-\alpha}, \quad K_{t+1} = (1-\delta) K_t + I_t.$$

$$\ln(\theta_t) = \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta,t}, \quad \ln(\lambda_t) = \rho_\lambda \ln(\lambda_{t-1}) + \varepsilon_{\lambda,t}$$

$$\beta = 0.99, \eta = 4, \alpha = 0.3, \delta = 0.025, \rho_\theta = \rho_\lambda = 0.99, \sigma = 10,$$

Big shocks variant: $\sigma_\theta = 0.20, \ \sigma_\lambda = 0.01.$

Small shocks variant: $\sigma_\theta = 0.01, \ \sigma_\lambda = 0.0005.$

RBC model: predicted standard deviations (HP filtered variables)

	Y	*C*	*I*	*K*	*N*	θ	λ
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*
(a) Model variant with big shocks (σ_θ=0.20, σ_λ=0.01)							
Second-order model approximation							
Both shocks	0.469	0.053	1.962	0.115	0.688	0.259	0.013
Just θ shock	0.124	0.037	0.483	0.038	0.212	0.259	0.000
Just λ shock	0.420	0.034	1.706	0.104	0.608	0.000	0.013
Linearized model							
Both shocks	0.229	0.041	1.059	0.095	0.350	0.259	0.013
Just θ shock	0.118	0.037	0.416	0.037	0.205	0.259	0.000
Just λ shock	0.196	0.016	0.974	0.087	0.284	0.000	0.013
(b) Model variant with small shocks (σ_θ=0.01, σ_λ=0.0005)							
Both shocks	0.011	0.002	0.053	0.005	0.018	0.013	0.001
Just θ shock	0.006	0.002	0.021	0.002	0.010	0.013	0.000
Just λ shock	0.010	0.001	0.049	0.004	0.014	0.000	0.001
Linearized model							
Both shocks	0.011	0.002	0.053	0.005	0.018	0.013	0.001
Just θ shock	0.006	0.002	0.021	0.002	0.010	0.013	0.000
Just λ shock	0.010	0.001	0.049	0.004	0.014	0.000	0.001

2^{nd}-order accurate RBC model: accuracy of filters (50 simulation runs with T=100 periods)

	ALL	Y	C	I	K	N	θ	λ
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(a) Model variant with big shocks (σ_θ=0.20, σ_λ=0.01)								
Average RMSEs								
Quad.Kalm.	0.176	0.039	0.006	0.039	0.435	0.039	0.141	0.023
Particle Filt.	0.597	0.527	0.108	0.499	0.892	0.695	0.658	0.030
Lin. Kalman	1.917	1.448	0.176	2.063	3.955	0.973	0.975	0.067
Fraction of runs in which RMSE is lower for KalmanQ than for other filters								
Particle Filt.	1.00	1.00	1.00	1.00	0.84	1.00	1.00	0.68
Lin. Kalman	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.98
(b) Model variant with small shocks (σ_θ=0.01, σ_λ=0.0005)								
Average RMSEs								
Quad.Kalm.	.0042	.0007	.0003	.0019	.0099	.0019	.0035	.0002
Particle Filt.	.0223	.0046	.0039	.0074	.0398	.0278	.0271	.0010
Lin. Kalman	.0508	.0224	.0078	.0819	.0716	.0466	.0357	.0019
Fraction of runs in which RMSE is lower for KalmanQ than for other filters								
Particle Filt.	1.00	1.00	1.00	1.00	0.90	1.00	1.00	0.92
Lin. Kalman	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.90

'Quad.Kalm.': Quadratic K. filter; 'Particle Filt.': Particle Filter (500,000 particles); 'Lin. Kalman': standard K. filter

Observables: Y,C,I,L (in logs); i.i.d. measurement error (std. 0.04 [0.002] in big [small] shocks variants.)

Computing time (filtering series T=100)—Quad. Kalman: 0.03 sec.; Particle Filter: 81.21 sec.; Linear Kalman: 0.01 sec.

APPLICATION TO 'REAL-WORLD' DATA

Kollmann (JMCB, 2013) estimated LINEARIZED two-country DSGE model with global banks, using quarterly data for US and EA (1990-2010).

Here will estimate a second-order approximation of the same model, using the same data.

[Large model: 19 state variables. Estimation of 3rd order approx. not feasible.]

International RBC model with Global Bank (Kollmann, Enders & Mueller, EER, 2011)

- Bank deposits from Home (H), Foreign (F) households,
- makes loans to H,F entrepreneurs
- Capital requirement
- Lending rate spread: decreasing function of bank capital

►**BANK CAPITAL CHANNEL that is representative of recent models**

● **Bank**

Assets (loans) and deposits (end of period t): L_{t+1}, D_{t+1}.

Bank equity: $E_t \equiv L_{t+1} - D_{t+1}$

Bank capital requirement: $L_{t+1} - D_{t+1} \geq \gamma_t L_{t+1}$

γ_t**: 'target' (benchmark) bank capital ratio**

Inequality constraints technically difficult.

Seems plausible that banks can, to some extent, circumvent capital requirement, but this is costly.

PENALTY FUNCTION

Excess capital: $x_t \equiv L_{t+1} - D_{t+1} - \gamma L_{t+1}$

Bank bears convex cost $\phi(x_t)$,

$$\phi(0)=0, \ \ \phi'(x_t)<0, \phi''(x_t)\geq 0$$

Bank can choose $L_{t+1} - D_{t+1} < \gamma L_{t+1}$ **but this is expensive**

Bank decision problem:

Max $E_0 \sum_{t=0}^{\infty} \beta^t \log(d_t^B)$ **s.t.**

$$L_{t+1} + D_t R_t^D + \phi(L_{t+1}(1-\gamma) - D_{t+1}) + d_t^B = L_t R_t^L - \Delta_t + D_{t+1}$$

$R_t^L [R_t^D]$ **: loan [deposit] rate (t-1 to t);** Δ_t**: loan default**

First order conditions for bank:

- $R_{t+1}^D E_{t+1} \beta d_t^B / d_{t+1}^B = 1 + \phi_t'$;
- $R_{t+1}^L E_{t+1} \beta d_t^B / d_{t+1}^B = 1 + (1-\gamma_t)\phi_t'$,

$$\Rightarrow R_{t+1}^L - R_{t+1}^D \cong -\gamma_t \phi'(L_{t+1}^W (1-\gamma_t) - D_{t+1}^W) > 0$$

Loan rate spread = marginal cost of excess leverage

Spread = f(excess bank capital), f’<0

Interesting non-linearity

Spread = f(excess bank capital), f'<0

f''<0 or f''>0 ?

- 11 Exogenous Shocks:

'Conventional' macro shocks (in H&F): TFP, investment efficiency, preference shocks (labor supply), government purchases

'Banking shocks':

1) H,F loan losses

2) shocks to required capital ratio (fraction of bank assets that has to be funded using bank's own funds)

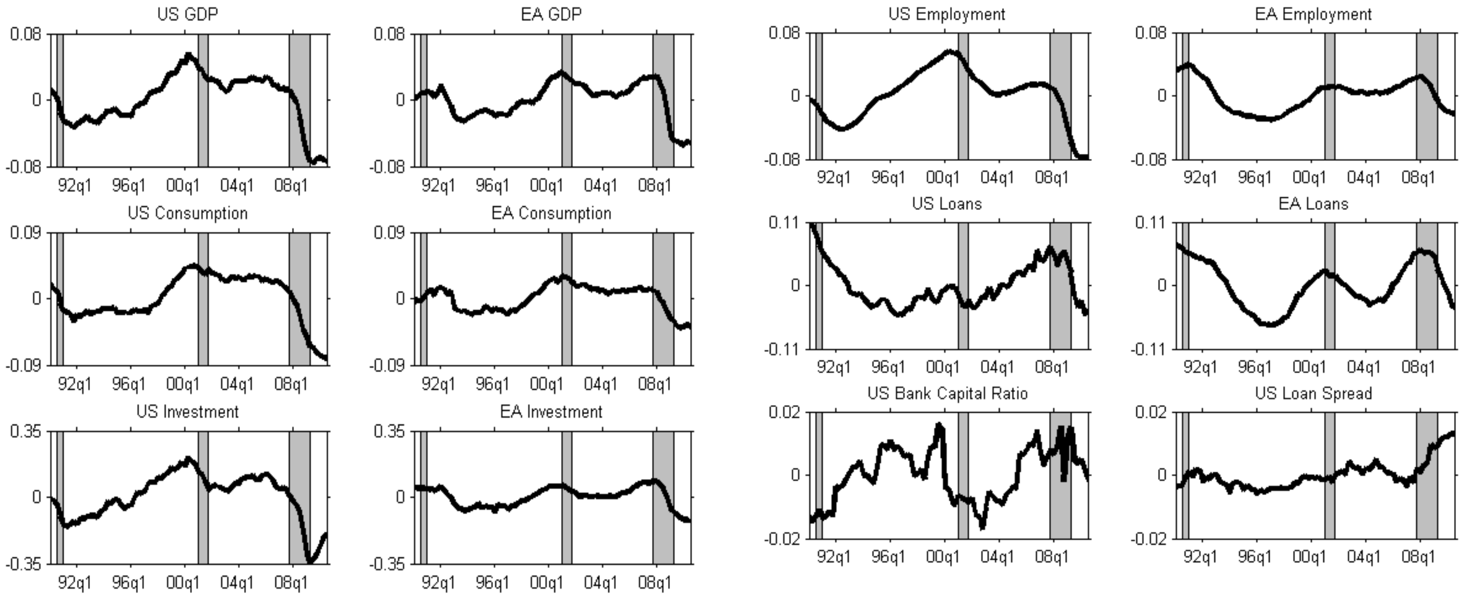
►DATA : US and Euro Area (EA), 1990-2010 (quarterly), macro & banking data

● Estimation uses linearly detrended data (logs), 12 observables

■ US & EA Y,C,I, Employment

■ Bank loans; bank capital ratio (bank equity/assets); loan rate spread

● allow for measurement error in all observables



Time series used in estimation

Posterior estimates of selected parameters

	Linearized Model		2nd order approx.	
Parameter	**Mean**	**Std**	**Mean**	**Std**
Slope **Spread** wrt Bnk CapRatio	-0.20	0.04	0.11	0.02
Second deriv. of **Spread**	**--**	**--**	-0.007	0.004
% Std of shock innovations				
US loan default/GDP	0.67	0.11	0.57	0.11
EA loan default/GDP	0.75	0.10	0.82	0.07
Benchmark Bank Cap Ratio	0.59	0.10	0.95	0.17
% Std of measurement error				
Loan Spread	0.02	0.00	0.02	0.00
US Loans	0.86	0.09	0.82	0.07
EA Loans	0.47	0.06	0.43	0.04
Bank Capital Ratio	0.43	0.06	0.33	0.03

Moments of HP filtered variables

	GDP		Investment		Employment	
	US	EA	US	EA	US	EA
Standard deviations (%)						
Data	1.12	1.14	5.08	2.87	1.15	0.70
Lin. Model	1.14	1.22	4.58	2.47	1.13	1.22
Quadratic Model	1.15	1.15	3.72	3.19	1.17	1.13
3rd-order Model	2.00	2.03	8.40	8.64	2.36	2.35 params of quadr model
Variance shares accounted for by banking shocks						
Linearized model						
Bank Shocks	3.60	4.23	9.98	25.19	7.11	8.22
2nd order approximated model						
Bank Shocks	7.25	11.82	28.71	38.44	13.07	21.15
Non-Bk Shocks	90.4	85.48	64.61	54.01	83.41	74.62
Interaction	2.70	2.68	6.67	7.54	3.51	4.22

% change in macro aggregates (relative to trend), '07q4-'09q2 recession (peak to trough)

	GDP		Investment		Employment	
	US	EA	US	EA	US	EA
DATA:	**-8.53**	**-7.49**	**-35.15**	**-15.94**	**-6.84**	**-2.82**

Share of drop due to banking shocks

	GDP US	GDP EA	Investment US	Investment EA	Employment US	Employment EA
Linearized model	**11.7**	**15.7**	**15.2**	**34.8**	**18.9**	**55.7**
Quadratic model	**18.9**	**29.8**	**33.6**	**82.1**	**27.5**	**92.9**

Quadratic model: state-contingent responses to exogenous innovations

GDP Responses to 1% TFP innovations, 1%GDP loan losses, 1ppt shock to target bank capital ratio

	TFP shock US	TFP shock EA	Loan loss US	Loan loss EA	Target bank capital ratio
Mean impact responses					
US GDP	1.43	-0.23	-0.05	-0.19	-0.09
EA GDP	-0.29	1.35	-0.07	-0.27	-0.10
Standard dev of impact responses					
US GDP	0.15	0.11	0.01	0.09	0.05
EA GDP	0.14	0.15	0.01	0.08	0.09
Correlations of impact responses with loan rate spread $(R_L - R_D)$					
US GDP	-0.36	0.42	0.35	0.37	-0.29
EA GDP	0.41	-0.34	0.09	0.35	-0.40

CONCLUSION

Developed tractable method for taking higher order approximated DSGE models to data, using likelihood-based approach

Showed that higher-order approximations increases estimated role of financial shocks for business cycle & GFC

Outlook for future work:

Revisit other estimated DSGE models

Explore other non-linearities (eg downward price/wage rigidity, asymmetric investment adjustment costs)

Estimation of (smaller) models approximated to third order

INCOMPLETE
(WORK IN PROGRESS)

Tractable Likelihood-Based Estimation of Non-Linear DSGE Models Using Higher-Order Approximations

Robert Kollmann (*)
ECARES, Université Libre de Bruxelles & CEPR
June 14, 2015

This paper discusses a tractable approach for computing the likelihood function of non-linear Dynamic Stochastic General Equilibrium (DSGE) models that are solved using second- and third order accurate approximations. By contrast to particle filters, no stochastic simulations are needed for the method here. The method here is, hence, much faster and it is thus suitable for the estimation of medium-scale models. The method assumes that the number of exogenous innovations equals the number of observables. Given an assumed vector of initial states, the exogenous innovations can thus recursively be inferred from the observables. This easily allows to compute the likelihood function. Initial states and model parameters are estimated by maximizing the likelihood function. Numerical examples suggest that the method provides reliable estimates of model parameters and of latent state variables, even for highly non-linear economies with big shocks.

- -

(*) I thank Matteo Iacoviello and Johannes Pfeifer for useful discussions. Thank are also due to Johannes Pfeifer for advice about the Dynare software. Financial support from Académie Universitaire Wallonie-Bruxelles (Action de recherche concertée, grant ARC-AUWB/2010-15/ULB-11) and from the European Commission (FP7 program; Macfinrobods, grant 612796) is gratefully acknowledged.
Address: ECARES, CP 114, ULB; 50 Av. F. Roosevelt, B-1050 Brussels, Belgium
robert_kollmann@yahoo.com

1. Introduction

During the last three decades, Dynamic Stochastic General Equilibrium (DSGE) models have become the workhorse of modern macroeconomic research. These models have also proven to be invaluable tools for policy analysis and economic forecasting. Due to their complexity, numerical approximations are required to solve DSGE models. The bulk of DSGE-based analysis uses linear approximations. A fast growing recent literature has taken *linearized* DSGE models to the data, using likelihood-based methods (early contributions include Kim (2000), Schorfheide (2000) and Otrok (2001)).

Linearity (in state variables) greatly facilitates model estimation, as it allows to use the standard Kalman filter to infer latent variables and to compute sample likelihood functions based on prediction error decompositions. However, linear approximations are inadequate for models with big shocks, and they cannot capture the effect of risk on economic decisions and welfare. Non-linear approximations are thus, for example, needed for welfare calculations in stochastic models, or for studying asset pricing and non-linearities due to financial frictions and constraints.

Recent research has begun to estimate *non-linear* DSGE models. That work has mainly used particle filters, i.e. filters that infer latent states using Monte Carlo methods (see Fernández-Villaverde and Rubio-Ramírez (2007) and An and Schorfheide (2007) for early applications). Particle filters are slow computationally, which limits their use to small models. Other attempts at empirical estimation of non-linear DSGE models use approximate deterministic filters, essentially non-linear versions of the Kalman filter. See, e.g., Ivashchenko (2014) and Kollmann (2015a) who present 'quadratic' filters for second-order approximate DSGE models; those filters are, however, based on the assumption that the residuals of second-order equated model equations are Gaussian. Nevertheless, these filters may be more accurate than particle filters, and they are clearly much faster than particle filters.

Guerrieri and Iacoviello (2014) point out that if initial values of the state variables are (assumed) known, then one can recursively infer the value of innovations in all periods from the observable data (conditional on the initial state), if the number of observables equals the number of shocks. This makes it unnecessary to use filters, and the likelihood function can easily be computed. Guerrieri and Iacoviello (2014) apply this idea to a simple DSGE model with an occasionally binding collateral constraint (all other model equations are linear), assuming that the initial state vector equals the steady state.

The paper here uses this insight to estimate DSGE models that are solved by second- or third- order Taylor expansions of the decision rules in the neighborhood of a deterministic steady state. 'Local' higher-order approximations of the type considered here are the most widely used non-linear solution methods for DSGE models; due to their great simplicity and speed, they are also currently the only usable non-linear solution methods for medium- scale models (see survey by Kollmann, Maliar, Malin and Pichler (2011) and Kollmann, Kim and Kim (2011)).[1] For this reason, it is important to develop a tractable method that allows *estimating* higher order approximated models. This paper focuses on the estimation of third-order approximated models. The method here can also easily be used for the estimation of second-order accurate models or for models of fourth (or higher) order of accuracy.[2]

A key problem in estimating second- and third order accurate models is that the decision rules include polynomials in the innovations to exogenous variables. Given the predetermined and exogenous variables realized at date t-1, multiple date t exogenous innovations are thus consistent with the period t observables. To overcome this problem, I consider restricted third-order date t decision rules that are *linear* in the date t exogenous innovations—the coefficients of those innovations may, however, be functions of lagged state variables. I show that these restricted decision rules are observationally indistinguishable from decision rules that include higher-order powers of contemporaneous exogenous innovations. Estimating the DSGE model with the restricted decision rules is straightforward. Numerical examples show that the estimation method here is both fast and accurate, even for models with strong non-linearity and big shocks.

While Guerrieri and Iacoviello (2014) postulate that the initial state equals the steady state, I *estimate* the initial state variables (together with the structural model parameters). This allows more precise estimation of latent state variables (in the estimation sample) and of the structural model parameters. In generic DSGE models, the state variables are highly persistent. Erroneously *assuming* that the initial state equals the steady state may thus induce large and persistent estimation errors for states in subsequent periods.

[1]Computer code that allows to easily implement the approximation methods is freely available; see, e.g., Chris Sims' (2000) gensys2 code, Schmitt-Grohé and Uribe's (2004) code, and the Dynare code of Adjemian et al. (2014).
[2] Second-order accurate models have, for example, proven useful for welfare analysis (e.g., Kollmann (2002, 2004). Third-order accurate models are needed to capture endogenous fluctuations in risk-premia.

2. Model format

Standard DSGE models can be expressed as:

$$E_t M(X_{t+1}, Y_{t+1}, X_t, Y_t, \varepsilon_{t+1}) = 0,$$

where E_t is the mathematical expectation conditional on date *t* information; $M: R^{2n+m} \to R^n$ is a function, and X_t is an n_Xx1 vector of exogenous variables and endogenous predetermined variables, while Y_t is an n_Yx1 vector of non-predetermined variables. ε_{t+1} is an *m*x1 vector of serially independent innovations to exogenous variables. In what follows, ε_t is Gaussian: $\varepsilon_t \sim N(0, \xi^2 \Sigma_\varepsilon)$, where ξ is a scalar that indexes the size of shocks. I assume that $n \equiv n_X + n_Y \geq m$.

The solution of model (1) is given by 'decision rules' $X_{t+1} = G(X_t, \varepsilon_{t+1}, \xi)$ and $Y_t = H(X_t, \xi)$ such that $E_t M(G(X_t, \varepsilon_{t+1}, \xi), H(G(X_t, \varepsilon_{t+1}, \xi), \xi), X_t, H(x_t, \xi), \varepsilon_{t+1}) = 0 \; \forall X_t$. See, e.g., Sims (2010), and Schmitt-Grohé and Uribe (2004) (who also show how generic DSGE models can be expressed in format (1)). Stacking the decision rules, we have $\Omega_{t+1} = F(X_t, \varepsilon_{t+1}, \xi)$, where Ω_{t+1} is the column vector $\Omega_{t+1} \equiv (X_{t+1}; Y_{t+1})$. This paper considers first-, second- and third-order accurate model solutions, namely first-, second- and third-order Taylor series expansions of the policy function around a deterministic stead state, i.e. around $\xi = 0$ and vectors X, Y such that $X = F(X, 0, 0)$, $Y = G(X, 0)$ and $\Omega = H(\Omega, 0, 0)$. Let $x_t \equiv X_t - X$, $y_t \equiv Y_t - Y$ and $\omega_t = (x_t; y_t)$.

First-, second- and third-order accurate model solutions have the following form:

$$\omega_{t+1} = F_1 x_t + F_2 \varepsilon_{t+1}, \tag{1}$$

$$\omega_{t+1} = F_0 \xi^2 + F_1 x_t + F_2 \varepsilon_{t+1} + F_{11} x_t \otimes x_t + F_{12} x_t \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \tag{2}$$

and $$\omega_{t+1} = F_0 \xi^2 + (F_1 + F_{1\xi} \xi^2) x_t + (F_2 + F_{2\xi} \xi^2) \varepsilon_{t+1} + F_{11} x_t \otimes x_t + F_{12} x_t \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} + \ldots$$

$$F_{111} x_t \otimes x_t \otimes x_t + F_{112} x_t \otimes x_t \otimes \varepsilon_{t+1} + F_{122} x_t \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} + F_{222} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \tag{3}$$

respectively. $F_0, F_1, F_{1s}, F_2, F_{2s}, F_{11}, F_{12}, F_{22}, F_{111}, F_{112}, F_{122}$ and F_{222} are matrices that are functions of the structural model parameters (i.e. parameters that describe preferences, technologies and other aspects of the economic environment). These matrices do not depend on the scale of shocks (ξ). $\otimes$ denotes the Kronecker product.

When simulating higher-order models it is common to use the 'pruning' scheme of Kim, Kim, Schaumburg and Sims (2008), under which products of state variables are replaced by products of variables approximated to lower order. Let $a_t^{(i)}$ denote a variable a_t approximated to *i*-th order. Under the pruning scheme, $(a_t b_t)^{(2)}$ is replaced by $a_t^{(1)} b_t^{(1)}$, $(a_t b_t)^{(3)}$ is replaced by $a_t^{(1)} b_t^{(2)} + a_t^{(2)} b_t^{(1)} - a_t^{(1)} b_t^{(1)} = a_t^{(2)} b_t^{(1)} + a_t^{(1)}(b_t^{(2)} - b_t^{(1)})$, and $a_t^{(1)} b_t^{(1)} c_t^{(1)}$ is replaced by $a_t^{(1)} b_t^{(1)} c_t^{(1)}$.

With pruning, the second-order solution (2) is, thus replaced by:

$$\omega_{t+1}^{(2)} = F_0 \xi^2 + F_1 x_t^{(2)} + F_2 \varepsilon_{t+1} + F_{11} x_t^{(1)} \otimes x_t^{(1)} + F_{12} x_t^{(1)} \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \text{ with } \omega_{t+1}^{(1)} = F_1 x_t^{(1)} + F_2 \varepsilon_{t+1}. \quad (4)$$

The pruned third-order solution is:

$$\omega_{t+1}^{(3)} = F_0 \xi^2 + F_1 x_t^{(3)} + F_{1\xi} \xi^2 x_t^{(1)} + (F_2 + F_{2\xi} \xi^2) \varepsilon_{t+1} + F_{11} \{x_t^{(2)} \otimes x_t^{(1)} + x_t^{(1)} \otimes (x_t^{(2)} - x_t^{(1)})\} + F_{12} x_t^{(2)} \otimes \varepsilon_{t+1} + F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} + \ldots$$
$$F_{112} x_t^{(1)} \otimes x_t^{(1)} \otimes \varepsilon_{t+1} + F_{122} x_t^{(1)} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} + F_{222} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1}. \quad (5)$$

Unless the pruning algorithm is used, second-order approximated models often generate exploding simulated time paths. Pruning ensures that higher-order accurate model solutions are non-explosive if the first-order system (1) is stationary (i.e. when all eigenvalues of F_1 are smaller than unity in absolute value).

The motivation for pruning is that, in repeated applications of (2), third and higher-order terms of state variables appear; e.g., when ω_{t+1} is quadratic in ω_t, then ω_{t+2} is quartic in ω_t; pruning removes these higher-order terms. The unpruned systems (2) and (3) have extraneous steady states (not present in the original model)--some of these steady states mark transitions to unstable behavior. Large shocks can thus move the model into an unstable region. Pruning overcomes this problem.

3. Inferring the exogenous innovations from observables

Assume that, at date t, the econometrician knows the state vectors $x_t^{(1)}, x_t^{(2)}, x_t^{(3)}$ and that she observed 'm' of the elements of the vector $\omega_{t+1}^{(3)}$ (or 'm' linear combinations of the elements of $\omega_{t+1}^{(3)}$), i.e. a vector $z_{t+1} \equiv Q \omega_{t+1}^{(3)}$, where Q is a known matrix of dimension mxn. (Recall that 'm' is the number of exogenous innovations.) (5) implies:

$$z_{t+1} = \gamma_t + Q(F_2 + F_{2\xi} \xi^2) \varepsilon_{t+1} + Q F_{12} x_t^{(2)} \otimes \varepsilon_{t+1} + Q F_{22} \varepsilon_{t+1} \otimes \varepsilon_{t+1} + \ldots$$
$$Q F_{111} x_t^{(1)} \otimes x_t^{(1)} \otimes x_t^{(1)} + Q F_{112} x_t^{(1)} \otimes x_t^{(1)} \otimes \varepsilon_{t+1} + Q F_{122} x_t^{(1)} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1} + Q F_{222} \varepsilon_{t+1} \otimes \varepsilon_{t+1} \otimes \varepsilon_{t+1}, \quad (6)$$

where $\gamma_t \equiv Q \cdot [F_0 \xi^2 + F_1 x_t^{(3)} + F_{1\xi} \xi^2 x_t^{(1)} + F_{11} \{x_t^{(2)} \otimes x_t^{(1)} + x_t^{(1)} \otimes (x_t^{(2)} - x_t^{(1)})\}]$ is a known quantity. As the right-hand side of (6) includes second and third powers of ε_{t+1}, one cannot uniquely solve (6) for the unknown vector of innovations ε_{t+1}. There does not appear to exist a tractable method for computing all of the vectors ε_{t+1} that solve (6) when m is larger than 2 or 3.

One approach to infer the 'true' ε_{t+1} might be to solve (6) for ε_{t+1} using a non-linear equation solve such as Chris Sims' csolve program, using $\varepsilon_{t+1}=0$ as an initial guess.[3] Experiments with a range of models suggest that when the variance of the true innovations is small, then this method detects the true ε_{t+1}.[4] However, this method is not reliable when shocks are large. Computationally, it is also relatively slow.

To avoid these complications, I abstract from the terms in $\varepsilon_{t+1}\otimes\varepsilon_{t+1}$ and in $\varepsilon_{t+1}\otimes\varepsilon_{t+1}\otimes\varepsilon_{t+1}$ in (5), and I consider the following 'restricted' third-order decision rule:

$$\omega_{t+1}^{(3)}=F_0\xi^2+F_1x_t^{(3)}+F_{1\xi}\xi^2x_t^{(1)}+(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+F_{11}\{x_t^{(2)}\otimes x_t^{(1)}+x_t^{(1)}\otimes(x_t^{(2)}-x_t^{(1)})\}+F_{111}x_t^{(1)}\otimes x_t^{(1)}\otimes x_t^{(1)}+\ldots$$

$$F_{12}x_t^{(2)}\otimes\varepsilon_{t+1}+F_{112}x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1}. \qquad (7)$$

Experiments with several models suggest that the restricted decision rule (7) is observationally almost indistinguishable from the third-order model (5), and that even for economies with strong curvature and big shocks. Simulating the decision rules (5) and (7) (using the same initial conditions and the same sequences of innovations) generates sequences of endogenous variables that are extremely highly correlated across (5) and (7) (see below).

Henceforth, I assume that the **true** data generating process is given by equations (4) and (7).

Note that when (7) is assumed, then the observation equation is given by:

$$z_{t+1}=\gamma_t+Q(F_2+F_{2\xi}\xi^2)\varepsilon_{t+1}+QF_{12}x_t^{(2)}\otimes\varepsilon_{t+1}+QF_{112}x_t^{(1)}\otimes x_t^{(1)}\otimes\varepsilon_{t+1}. \qquad (8)$$

This expression is linear in ε_{t+1}. It can be written as $z_{t+1}=\gamma_t+\lambda_t\varepsilon_{t+1}$ where λ_t is an (m *x* m) matrix. Provided that λ_t is non-singular, one can thus infer ε_{t+1} from date t+1 observables:

$$\varepsilon_{t+1}=(\lambda_t)^{-1}(z_{t+1}-\gamma_t). \qquad (9)$$

[3] I thank Matteo Iacoviello for suggesting this approach to me.

[4] I simulate various models by feeding a sequence $\{\varepsilon_{t+1}\}$ into (4),(5); I then tried to infer the innovations from the observables by solving (6) for ε_{t+1} using the csolve algorithm. When the 'true' innovations are small, the method recovers the true values.

[Other approaches: Assume that ε_{t+1} is a discrete random variable that takes a finite number of values. The unrestricted third-order equation (6) has a unique solution, in the set of admissible values of ε_{t+1}. Disadvantage: when fitting the model to real world data (not simulated data), then (6) cannot hold exactly, when ε_{t+1} is discrete. Thus one has to allow for measurement error. Then number of shocks + measurement errors exceeds number of observables. Then cannot exactly recover shocks + measurement error from observables, and thus need a (particle) filter. Also, ε_{t+1} that generates smallest measurement error in (6) is a discontinuous function of the initial state vector and of the model parameters. Thus the likelihood function too is discontinuous in the initial state.]

4. Sample likelihood

Given the initial state $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}$ and data $\{z_t\}_{t=1}^T$ one can recursively compute the innovations $\{\varepsilon_t\}_{t=1}^T$ and the states $\{x_t^{(i)}, y_t^{(i)}\}_{t=1}^T$ for i=1,2,3 using (4),(7) and (9). The log likelihood of the data, conditional on $x_0^{(1)}, x_0^{(2)}, x_0^{(3)}$ is:

$$\ln L(\{z_t\}_{t=1}^T | x_0^{(1)}, x_0^{(2)}, x_0^{(3)}) = -(mT/2)\ln(2\pi) - (T/2)\ln|\xi^2\Sigma_\varepsilon| - \sum_{t=1}^T \{\varepsilon_t' (\xi^2\Sigma_\varepsilon)^{-1}\varepsilon_t + \ln|\lambda_{t-1}|. \qquad (10)$$

One can estimate the initial state, and the structural model parameters, by maximizing the likelihood function with respect to the initial states and parameters.

5. Application I: basic RBC model

I now illustrate the method for the basic RBC model. Assume a closed economy with a representative infinitely-lived household whose date t expected lifetime utility V_t is given by $V_t=\{\frac{1}{1-\sigma}C_t^{1-\sigma}-\frac{1}{1+1/\eta}\psi_t N_t^{1+1/\eta}\}+\lambda_t\beta E_t V_{t+1}$, where C_t and N_t are consumption and hours worked, at t, respectively. $\sigma>0$ and $\eta>0$ are the risk aversion coefficient and the (Frisch) labor supply elasticity. $0<\beta<1$ is the steady state subjective discount factor. $\psi_t>0$ and $\lambda_t>0$ are exogenous preference shocks: ψ_t is a labor supply shock, while λ_t is a shock to the subjective discount factor. ψ_t and λ_t equal unity in steady state. The household maximizes expected lifetime utility subject to the period t resource constraint

$$C_t+I_t+G_t=Y_t,$$

where Y_t and I_t are output, gross investment and exogenous government consumption, respectively. The production function is

$$Y_t = \theta_t K_t^{\alpha} N_t^{1-\alpha}$$

where K_t is the beginning-of-period t capital stock, and $\theta_t > 0$ is exogenous total factor productivity (TFP). The law of motion of the capital stock is

$$K_{t+1} = (1-\delta)K_t + I_t.$$

$0 < \alpha, \delta < 1$ are the capital share and the capital depreciation rate, respectively. The household's first-order conditions are:

$$\lambda_t E_t \beta (C_{t+1}/C_t)^{-\sigma} (\theta_{t+1} \alpha K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta) = 1, \quad C_t^{-\sigma}(1-\alpha)\theta_t K_t^{\alpha} N_t^{-\alpha} = \psi_t N_t^{1/\eta}.$$

The forcing variables follow independent autoregressive processes:

$$\ln(\theta_t/\theta) = \rho_\theta \ln(\theta_{t-1}/\theta) + \varepsilon_{\theta,t},\ \ln(G_t/G) = \rho_G \ln(G_{t-1}/G) + \varepsilon_{G,t},\ \ln(\psi_t) = \rho_\psi \ln(\psi_{t-1}) + \varepsilon_{\psi,t},\ \ln(\lambda_t) = \rho_\lambda \ln(\lambda_{t-1}) + \varepsilon_{\lambda,t},$$

with $0 < \rho_\theta, \rho_G, \rho_\Psi, \rho_\lambda < 1,$ where θ and G are steady state TFP and steady state government purchases. $\varepsilon_{\theta,t}, \varepsilon_{G,t}, \varepsilon_{\psi,t}$ and $\varepsilon_{\lambda,t}$ are normal i.i.d. white noises with standard deviations $\sigma_\theta, \sigma_G, \sigma_\psi$ and σ_λ.

The numerical simulations discussed below assume $\beta = 0.99, \eta = 4, \alpha = 0.3, \delta = 0.025;$ the steady state ratio of government purchases to GDP (G/Y) is set at 0.2. The autocorrelations of all forcing variables is set at $\rho_\theta = \rho_G = \rho_\psi = \rho_\lambda = 0.99,$ i.e. these exogenous variables undergo persistent fluctuations. These parameter values in that range are standard in (quarterly) macro models. The risk aversion coefficient is set at a high value, $\sigma = 10,$ so that the model has enough curvature to allow for non-negligible differences between the second- and third-order model approximations and the and linearized model. In all model variants, I set the scalar ξ that indexes the size of shocks at $\xi = 1.$ One model variant, referred to as the 'small shocks' variant, assumes $\sigma_\theta = \sigma_G = \sigma_\psi = 1\%$ and $\sigma_\lambda = 0.025\%$. Those shock sizes (i.e. rate of time preference shocks 40-times smaller than the other shocks) ensure that each shock accounts for a non-negligible share of the variance of the endogenous variables (see Table 1). That 'small shocks' calibration is standard in the RBC literature, and it implies that the volatility of the endogenous variables in the model is roughly consistent with the empirical volatility. In the 'small shocks' variant, the behavior of endogenous variables predicted by the second- and third-order approximated model is broadly similar to that predicted by the linearized model. I thus also consider model variants with much bigger shocks—in those variants, the higher-order approximated model generates predicted behavior that differs noticeably from behavior in the first-order approximated model. In one model variant, I set the standard deviations or shocks 5 times greater than in the 'small shocks' variant $(\sigma_\theta = \sigma_G = \sigma_\psi = 5\%, \sigma_\lambda = 0.125\%)$; I also consider a variant in which the standard deviation of exogenous innovations is 10 time greater $(\sigma_\theta = \sigma_G = \sigma_\psi = 10\%, \sigma_\lambda = 0.250\%)$. I refer to these model variants as the 'big shocks' variant and the 'very big shocks' variant, respectively.

I solve the model using the Dynare toolbox (Adjemian et al. (2014)). The Taylor expansions of the model equations are taken with respect to logs of all variables.

5.1. Predicted standard deviations and mean values

Table 1 reports predicted standard deviations of GDP, consumption, investment, hours worked and the capital stock. All variables are expressed in logs. The predicted moments are shown for variables in levels, as well as for first-differenced variables. In the 'small shocks' variant, the order of approximation does not matter much for predicted behavior. For example, the predicted standard deviation of GDP is 3.00% (3.09%) [2.04%] under the first- (second-) [third-] order accurate model approximation.

By contrast, in the model variants with 'big' and with 'very big' shocks, the second- and third-order approximations generate markedly greater volatility of the endogenous variables than the linear approximation. In the 'big shocks' ['very big shocks'] variant the predicted volatility of GDP rises by one quarter [doubles] when the third-order approximation is used, instead of the linear approximation.

Under the linear approximation, the unconditional means of all endogenous variables equals their values in the deterministic steady state. Under the second- and third-order approximations, the unconditional means can differ from the steady state (unconditional means implied by the second and third-order approximations are identical). In the 'small shocks' variant, the mean of capital stock and mean GDP exceeds steady state values by 0.81% and 0.25%, respectively. This is due to precautionary saving that is captured by the second-order approximation. In the 'big shocks' ['very big shocks'] model variant, the mean capital stock and mean GDP are 20.39% and 6.26% [81.56% and 25.05%] above steady state.

5.2. Comparing the 'restricted' versions of the third-order accurate model

Table 2 documents that the 'restricted' version (7) of the (pruned) third-order accurate model is observationally equivalent to the 'unrestricted' version (5). The correlation between time series generated by these variants are very close to unity, for GDP, consumption, investment, hours and the capital stock (both in levels and in first differenced), and that even when shocks are very big.

5.3. Estimating structural parameters and the initial state

I now evaluate the ability of the estimation method to estimate structural model parameters and latent state variables. For each of the three model variants, I generated 40 simulation runs of 5100 periods (each simulation run was initiated at the unconditional means of the state variables). I use the last 100 periods of each simulation run for estimation. Estimation is conducted by maximizing the sum of the likelihood function (10) and a prior log pdf of the initial state (see below). I estimate the initial states and 10 structural parameters: the risk aversion coefficient (σ), labor supply elasticity (η), as well as the autocorrelations and standard deviations of the four exogenous variables. As the model has four exogenous shocks, four observables are needed for estimation. I use first differences of log GDP, consumption, investment and hours worked as observables.

The model has 5 state variables: the capital stock, and the lagged values of each of the four exogenous variables. The likelihood depends on the first-, second- and third- order accurate initial values of these 5 state variables (see (10)). The laws of motion of the four exogenous variables are log-linear. Hence, their values are identical under (log) approximations of orders 1,2 and 3. To reduce the computational burden, I assume (in the current version of the paper) that $k_0^{(2)}=k_0^{(3)}$ and $k_0^{(1)}=k_0^{(3)}-E(k_0^{(3)}-k_0^{(1)});$ in other terms, the second-order accurate initial state capital stock is assumed to equal the third-order accurate capital stock; the first-order accurate initial capital stock is assumed to equal to the third-order accurate initial capital stock, adjusted for the difference between the mean values of these capital stocks.

The precision of the estimates of the state variables and of the model parameters is higher if prior information about the mean and variance of the initial state is used. I use a multivariate normal prior for the initial state vector $(\ln K_0^{(3)}, \ln\theta_0^{(3)}, \ln G_0^{(3)}, \ln\psi_0^{(3)}, \ln\lambda_0^{(3)});$ the prior mean and covariance are set to the unconditional means implied by the third-order accurate model (7) and the unconditional covariance implied by the second-order accurate model (4).[5]

Panel (a) of Table 3 reports the mean, median and standard deviation of the estimated model parameters across the 40 simulation runs, for the 'small shocks' model variant (Columns (1)-(3)), the 'big shocks' variant (Cols. (4)-(6)) and the 'very big shocks' model variant (Cols. (7)-(9)). For each simulation run, I compute the correlation between each estimated state variables (implied by the estimates of structural model parameters) and the true state variables. Panel (b) of Table 3 reports the mean, median and standard deviation of the correlation, across the 40 simulation runs (for each model variant).

Table 3 shows that, for all three model variants, the risk aversion coefficient, the autocorrelations of the exogenous variables and the standard deviations of exogenous innovations are tightly estimated: the mean and median parameter estimates (across runs) are close to the true parameter values, and the standard deviations of the parameter estimates are small. The labor supply elasticity η is less tightly estimated, in the model variants with 'big shocks' and with 'very big shocks', the median estimates (across 40 runs) are close to the true value $(\eta=4)$, but the standard deviation of the estimates is sizable.

The estimation method provides remarkably accurate estimates of the 5 state variables. The estimates of the capital stock, TFP, government purchases and the labor supply shock (ψ) are essentially perfectly correlated with the true values of these states, and that irrespective of the size of the shocks. The shock to the rate of time preference (λ) is somewhat less precisely estimated; the median correlations between estimates and true values of λ are 0.98-0.99 (across simulation runs).

6. Application II: DSGE model with banks and capital requirements

[Estimation of the DSGE model with banks presented in Kollmann (2013)]

7. Application III: DSGE model with volatility shocks

[Estimation of the 'Risk Matters' model of Fernández-Villaverde et al. (2011) and Born and Pfeifer (2014)]

[5] I use the covariance of second-order accurate variables, as the latter can be computed using formulae in Kollmann (2015); future versions will use the unconditional variance of third-order accurate variables (formulae to be derived).

References (incomplete)

Adjemian, S., H. Bastani, M. Juillard, F. Mihoubi, G. Perendia, J. Pfeifer, M. Ratto, S. Villemot (2014). Dynare: reference manual, Version 4.4.3., Working Paper, CEPREMAP.

An, S. and F. Schorfheide, 2007. Bayesian Analysis of DSGE models. Econometric Reviews 26, 113–172.

Born, B. and J. Pfeifer, 2014. Risk Matters: The Real Effects of Volatility Shocks: Comment. American Economic Review 104, 4231-4239.

Fernández-Villaverde, J. and J. Rubio-Ramírez, 2007. Estimating Macroeconomic Models: a Likelihood Approach. Review of Economic Studies 74, 1059–1087.

Fernández-Villaverde, J., P. Guerrón-Quintana, J. Rubio-Ramírez and M. Uribe, 2011. Risk Matters: the Real Effects of Volatility Shocks. American Economic Review 101, 2530-2561.

Guerrieri, L. and M. Iacoviello (2014). Collateral Constraints and Macroeconomic Asymmetries. Working Paper, Federal Reserve Board.

Ivashchenko, S. 2014. DSGE Model Estimation on the Basis of Second-Order Approximation. Computational Economics 43, 73-81.

Kim, J., Kim, S., Schaumburg, E. and C. Sims, 2008. Calculating and Using Second-Order Accurate Solutions of Discrete-Time Dynamic Equilibrium Models, Journal of Economic Dynamics and Control 32, 3397-3414.

Kollmann, R., 2004. Solving Non-Linear Rational Expectations Models: Approximations based on Taylor Expansions, Working Paper, University of Paris XII.

Kollmann, R., S. Maliar, B. Malin and P. Pichler 2011. Comparison of Numerical Solutions to a Suite of Multi-Country Models. Journal of Economic Dynamics and Control 35, pp.186-202

Kollmann, R. 2013. Global Banks, Financial Shocks and International Business Cycles: Evidence from an Estimated Model. Journal of Money, Credit and Banking 45(S2), 159-195.

Kollmann, R. 2015a. Tractable Latent State Filtering for Non-Linear DSGE Models Using a Second-Order Approximation and Pruning, Computational Economics 45, 239-260.

Kollmann, R. 2015b. Tractable Latent State Filtering for Non-Linear DSGE Models Using a Third- Order Approximation and Pruning, work in progress.

Otrok, Christopher. 2001. On Measuring the Welfare Cost of Business Cycles. Journal of Monetary Economics 47, 61-92.

Schmitt-Grohé, S. and M. Uribe, 2004. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. Journal of Economic Dynamics and Control 28, 755 – 775.

Sims, C. 2000. Second Order Accurate Solution of Discrete Time Dynamic Equilibrium Models. Working Paper, Economics Department, Princeton University

Table 1. RBC model: predicted standard deviations (in%)

	Y	*C*	*I*	*N*	*K*
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with small shocks $(\sigma_\theta=\sigma_G=\sigma_\psi=0.01, \sigma_\lambda=0.00025)$					
(a.1) Variables in levels (logs)					
1st order, all shocks	3.00	1.46	10.35	10.46	7.80
1st order, just θ shock	2.08	1.37	6.21	9.43	4.58
1st order, just G shock	1.59	0.08	1.46	1.90	1.03
1st order, just ψ shock	1.08	0.70	3.26	0.93	2.32
1st order, just λ shock	1.68	0.22	7.60	1.55	5.81
2nd order, all shocks	3.09	1.46	10.40	10.45	7.82
3rd order, all shocks	3.04	1.46	10.45	10.44	7.86
(a.2) First-differenced variables (logs)					
1st order, all shocks	0.67	0.17	2.62	1.12	0.17
2nd order, all shocks	0.67	0.17	2.62	0.12	0.17
3rd order, all shocks	0.68	0.17	2.63	0.13	0.17
(b) Model variant with big shocks $(\sigma_\theta=\sigma_G=\sigma_\psi=0.05, \sigma_\lambda=0.00125)$					
(b.1) Variables in levels (logs)					
1st order, all shocks	14.99	7.33	51.76	52.31	39.05
2nd order, all shocks	15.89	7.32	53.87	52.29	39.07
3rd order, all shocks	18.71	7.33	60.18	51.62	44.98
(b.2) First-differenced variables (logs)					
1st order, all shocks	3.35	0.85	13.09	5.63	0.86
2nd order, all shocks	3.56	0.85	13.45	5.77	0.88
3rd order, all shocks	4.00	0.83	14.63	5.94	0.92
(c) Model variant with very big shocks $(\sigma_\theta=\sigma_G=\sigma_\psi=0.10, \sigma_\lambda=0.00250)$					
(c.1) Variables in levels (logs)					
1st order, all shocks	29.99	14.66	103.52	104.62	78.01
2nd order, all shocks	35.41	14.65	115.43	105.46	86.33
3rd order, all shocks	58.77	14.87	166.39	103.71	123.94
(c.2) First-differenced variables (logs)					
1st order, all shocks	6.71	1.70	26.19	11.27	1.72
2nd order, all shocks	7.92	1.71	29.08	12.33	1.86
3rd order, all shocks	11.54	1.63	39.34	14.80	2.21

Note: Standard deviations (std.) of logged variables (listed above Cols. (1)-(5)) are shown for the RBC model. All moments are computed based on one simulation run of 5000 periods (the run is initiated at the unconditional mean of the state variables). Rows labeled '1st order', '2nd order' and '3rd order' show standard deviations predicted by the first-, second- and third-order accurate model variants, respectively. *Y*: GDP; *C*: consumption; *I*: gross investment; *N*: hours worked; *K*: capital stock.

Table 2. RBC model: correlations between variables predicted by 'full' and 'restricted' versions of third-order accurate model (see (5), (7))

	Y	C	I	N	K
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
(a) Model variant with small shocks (σ_θ=σ_G=σ_ψ=0.01, σ_λ=0.00025)					
(a.1) Variables in levels (logs)					
All shocks	1.000	1.000	1.000	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(a.2) First-differenced variables (logs)					
All shocks	1.000	1.000	1.000	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(b) Model variant with big shocks (σ_θ=σ_G=σ_ψ=0.05, σ_λ=0.00125)					
(b.1) Variables in levels (logs)					
All shocks	1.000	1.000	1.000	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(b.2) First-differenced variables (logs)					
All shocks	1.000	1.000	0.996	1.000	1.000
Just θ shock	1.000	1.000	0.999	1.000	1.000
Just G shock	0.999	0.999	1.000	0.999	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	0.998	1.000	1.000
(c) Model variant with very big shocks (σ_θ=σ_G=σ_ψ=0.10, σ_λ=0.00250)					
(c.1) Variables in levels (logs)					
All shocks	1.000	1.000	0.999	1.000	1.000
Just θ shock	1.000	1.000	1.000	1.000	1.000
Just G shock	1.000	1.000	1.000	1.000	1.000
Just ψ shock	1.000	1.000	1.000	1.000	1.000
Just λ shock	1.000	1.000	1.000	1.000	1.000
(c.2) First-differenced variables (logs)					
All shocks	1.000	1.000	0.984	0.999	1.000
Just θ shock	1.000	1.000	0.995	1.000	1.000
Just G shock	0.998	0.998	0.998	0.998	1.000
Just ψ shock	1.000	1.000	0.998	1.000	1.000
Just λ shock	1.000	1.000	0.992	1.000	1.000

Note: Correlations between variables predicted by the 'full' and 'restricted' third-order models are reported. 'All shocks': simulations with all 4 shocks. 'Just θ shocks', 'Just G shocks' etc. pertain to simulations in which just one type of shock is fed into the model; the other exogenous variables are set at steady state values (model is solved assuming 4 shocks). Reported statistics are based on one simulation run of 5000 periods (the run is initiated at the unconditional mean of the state variables). Y: GDP; C: consumption; I: gross investment; N: hours worked; K: capital stock. Correlations greater than 0.9995 are reported as 1.000.

Table 3. RBC model: estimates of structural parameters and of state variables, 40 simulation runs (100 periods)

	Model variant with 'small shocks'			Model variant with 'big shocks'			Model variant with 'very big shocks'		
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*
(a) Parameter estimates									
	Mean	**Median**	**Std**	**Mean**	**Median**	**Std**	**Mean**	**Median**	**Std**
σ	9.97	9.96	0.55	10.06	9.97	0.79	10.44	10.31	1.27
η	4.47	4.04	1.38	7.05	4.24	7.88	4.74	3.59	5.54
ρ_θ	0.99	0.99	0.002	0.99	0.99	0.002	0.99	0.99	0.01
ρ_G	0.98	0.99	0.013	0.98	0.99	0.018	0.98	0.99	0.01
ρ_ψ	0.99	0.99	0.003	0.99	0.99	0.005	0.98	0.99	0.03
ρ_λ	0.99	0.99	0.006	0.99	0.99	0.009	0.99	0.99	0.01
σ_θ (%)	1.01	0.99	0.06	5.07	5.02	0.32	9.98	9.96	0.79
σ_G (%)	1.00	0.99	0.08	4.91	4.77	0.73	12.42	10.88	4.73
σ_ψ (%)	0.99	0.99	0.05	4.97	4.79	0.65	10.69	9.93	2.49
σ_λ (%)	0.025	0.025	0.000	0.13	0.12	0.01	0.25	0.25	0.02
(b) Correlation between estimated & true states									
	Mean	**Median**	**Std**	**Mean**	**Median**	**Std**	**Mean**	**Median**	**Std**
$\ln K$	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln \theta$	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln G$	1.00	1.00	1.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln \psi$	1.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00	0.00
$\ln \lambda$	0.88	0.98	0.28	0.95	0.99	0.20	0.94	0.99	0.16

Note: The Table summarizes estimation results across 40 simulation runs of 100 periods each. Panel (a) reports the mean, median and standard deviation of the estimated model parameters across the 40 runs, for the 'small shocks' model variant (Columns (1)-(3)), the 'big shocks' variant (Cols. (4)-(6)) and the 'very big shocks' variant (Cols. (7)-(9)). For each simulation run, the correlation between the estimated state variables and the true state variables was computed. Panel (b) reports the mean, median and standard deviation of that correlation, across the 40 simulation runs (for each model variant). Mean/median correlations above 0.9995 are reported as 1.00.

The ***true*** values of the estimated parameters are: σ=10, η=4, $\rho_\theta=\rho_G=\rho_\psi=\rho_\lambda$=0.99. In the 'small shocks' model variant, the true standard deviations of exogenous innovations are: $\sigma_\theta=\sigma_G=\sigma_\psi$=1%, σ_λ=0.025%. 'Big shocks' model variant: $\sigma_\theta=\sigma_G=\sigma_\psi$=5%, σ_λ=0.125%. 'Very big shocks' model variant: $\sigma_\theta=\sigma_G=\sigma_\psi$=10%, σ_λ=0.250%.