Linearization about the Current State: A Computational Method for Approximating Nonlinear Policy Functions during Simulation *

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Abstract

This paper presents an adjustment to commonly used approximation methods for dynamic stochastic general equilibrium (DSGE) models. Policy functions approximated around the steady state will be inaccurate away from the steady state. In some cases, this does not lead to substantial inaccuracies. In other cases, however, the model may not have a well-defined steady state, or the nature of the steady state may be at odds with its off-steady-state dynamics. We show how to simulate a DSGE model by approximating about the current state. Our method introduces an approximation error, but minimizes the error associated with a finite-order Taylor-series expansion of the models characterizing equations. This method is easily implemented using available simulation software and has the advantage of mimicking highly non-linear behavior. We illustrate this with a variety of simple models. We compare our technique with other simulation techniques and evaluate the relative size of the two approximation errors mentioned above.

keywords: dynamic stochastic general equilibrium, linearization methods, numerical simulation, computational techniques, simulation modeling.

JEL classifications: C63, C68, E37

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are an important class of macroeconomic modeling that have been in use now for over two decades. They are increasingly used in policy contexts to simulate the effects of policy changes on the macroeconomy.[1]

Usually these models are too complex to find closed-form solutions for dynamic policy functions that map the current state of the economy into the values for next period's endogenous state variables. Instead, these models must be solved and simulated using some approximation method. One common technique is to approximate the non-linear behavioral equations of the model as a quadratic objective function that is maximized subject to linear constraints. This is the technique employed in the seminar paper by Kydland and Prescott (1982). Another approach is to use a generalized version of the state-space solution method pioneered by Blanchard and Kahn. (1980). Approaches that are computationally more intensive include the use of discrete grids, sometimes sparse as in Krueger and Kubler (2004), and often smoothly fitted by polynomial approximations or splines. More commonly used techniques include the linearization methods in Uhlig (1999) and Christiano (2002) who employ a method of undetermined coefficients to solve the state-space representation outlined in Blanchard and Kahn. (1980). Higher-order polynomial approximations developed by Judd (1992), Guu and Judd (2001), Collard and Juillard (2001) and Schmitt-Grohe and Uribe (2004) are increasingly widely used. Judd et al. (2011) introduce a generalized stochastic simulation algorithm based on the parameterized expectations literature that has very nice properties and is able to handle large state spaces that other methods cannot.

This paper presents an easy adjustment to the latter linear and higher-order approximation methods. Since approximation is almost always done about the models steady state, the linear policy functions can be inaccurate if the simulation is often away from the steady state. In some cases, this leads to only small errors. In other

[1] see for example Smets and Wouters (2007) and Christiano et al. (2005).

cases, however, the model may not have a well-defined steady state, or the nature of the steady state may be at odds with its off-steady-state dynamics.

Near steady state dynamics can be very different from the dynamics away from the steady state when there are occasionally binding constraints, such as credit rationing or non-negativity constraints on capital. Bend points in tax rates are yet another case where the steady state dynamics can be inaccurate.

In addition, approximating about the steady state requires the existence of a steady state. If one does not exist, the models variables must be redefined so that they are stationary. Some models of interest to researchers cannot be easily transformed in this manner, however. These include multi-sector models with unbalanced growth and models where the parameters are time-varying.

We show how to simulate a DSGE model by approximating about the current state, rather than the steady state. This method is easily implemented and has the advantage of accurately mimicking non-linear behavior. We illustrate this with several simple examples.

We proceed similarly to Uhlig (1999) and Christiano (2002) who linearize their dynamic equations about the steady state. They hypothesize linear policy functions and use the method of undetermined coefficients to solve for the coefficients of the policy function. We show how linearizing about an arbitrary point other than the steady state results in a very similar problem. The only difference being the presence constant terms in both the linearized dynamic equations and the approximate policy function. One disadvantage of simulating using this approximate policy function is that it often converges to a value other than the actual steady state.

We propose an alternative strategy where one approximates the policy function for each period of the simulation about the current state of the economy. This is computationally more intensive than linearizing about the steady state as it requires linearization each period, rather than only once. However, this method has the advantage of being much more accurate when the economy is far from the steady state. This method can easily replicate the behavior of highly nonlinear policy functions. This is because the Taylor-series approximation is highly accurate in the neighborhood of the point about which a function is approximated. Since we are always approximating about the current state, our linear policy function will be very close to the true policy function for that state.

While we illustrate our method using linear approximations, the concepts and method apply to higher-order polynomial approximations as well. Regardless of the degree of the polynomial approximation, the accuracy will be better in the neighborhood of the linearization point. Whenever the simulation deviates far away from the steady state, a steady state linearization will become less accurate. Linearization about the current state eliminates this approximation inaccuracy, albeit at the cost of more computational intensity.

We compare the performance of various linearization techniques against more complicated approximation techniques. For our first example we consider a case where the functional form of the true policy function is known. In this case the policy function is log-linear. We show that linearizing about the current state performs just as well as linearizing about the steady state using measures of deviation of the time paths for the approximate policy functions versus the true policy function. Since steady state linearization is computationally less intensive than our method, this does not make a strong case for use of our technique.

However, when we consider more complicated, but still common, models we find that linearizing about the current state leads to better goodness of fit. In these cases, we do not know the true policy function, but we can closely approximate it by grid search or spline methods. These methods are very accurate, but are also much more computationally intensive. Many complicated, but interesting models cannot be simulated this way, but can be simulated by polynomial approximation methods. Our results suggest that approximating about the current state will yield more accurate simulations, especially in cases when the economy often deviates away from the steady state substantially.

Our exposition is organized as follows. Section 2 explains the basics of approximation about an arbitrary state of the economy. Section 3 discusses in detail how approximation can be done about the economy's current state. Our technique introduces a new approximation error, but minimizes errors from approximating behavior away from the steady state. Sections 4.1 through 4.3 illustrate our technique for a variety of simple models, some of which are poorly approximated by standard methods. We compare the accuracy and computational load of various techniques. Section 5 concludes.

2 Approximation about an Arbitrary State

Consider a set of nonlinear expectational functions, in our case from a dynamic general equilibrium model. The state variables are grouped into two categories: exogenous state variables are grouped into the $S \times 1$ column vector, Z_t, while endogenous ones are placed in the $K \times 1$ column vector, X_t. There are R equations and they can be represented as in equation (2.1).

$$E_t\{\Gamma(X_{t+1}, X_t, X_{t-1}, Z_{t+1}, Z_t\} = 0 \tag{2.1}$$

This system of equations can be approximated by taking a first-order Taylor series expansion about an arbitrary point in the state space, $\theta_0 = \{X_0, Z_0\}$. This transformation is given in equation (2.2).

$$E_t\{T + F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\} = 0 \tag{2.2}$$

In the above equation, F, G & H are $R \times S$ matrices, L & M are $R \times K$ matrices, and T is an $R \times 1$ vector. All these will depend on which point is chosen for the linearization. Tildes denote absolute deviations from θ_0 values. Note that if we choose to linearize about the steady state, $\bar{\theta} = \{\bar{X}, \bar{Z}\}$ the value of T is zero. While this is true of the steady state, it will not be true generally.

The law of motion for the exogenous state variables is assumed to be a first-order vector autoregression of the form in equation (2.3).

$$Z_{t+1} = (I - N)\bar{Z} + NZ_t + E_{t+1} \tag{2.3}$$

Since we are allowing for linearization around any value of Z, we proceed to transform (2.3) into (2.4).

$$E_t\{\tilde{Z}_{t+1}\} = N\tilde{Z}_t - (Z_0 - \bar{Z}) \tag{2.4}$$

As with standard linearization techniques, our goal is to find a linear approximation to the policy function, (2.5).

$$\tilde{X}_t = U + P\tilde{X}_{t-1} + Q\tilde{Z}_t \tag{2.5}$$

Where U is an $S \times 1$ column vector, P is an $S \times S$ matrix and Q is $S \times K$.

The major difference between (2.5) and the standard linear policy function, is the inclusion of the constant term, U. This makes it possible for the endogenous state variables to drift away from the current state. Iterative substitution of (2.4) and (2.5) into (2.2) yields (2.6).

$$\begin{aligned} &[(FP+G)P+H]\tilde{X}_{t-1} + [(FQ+L)N + (FP+G)Q + M]\tilde{Z}_t \\ &\quad + T + [F(I+P)+G]U + (FQ+L)(N-I)(Z_0-\bar{Z}) = 0 \end{aligned} \tag{2.6}$$

Equation (2.6) imposes three conditions on, P, Q & U which allows us to identify these matrices. The first condition is:

$$FP^2 + GP + H = 0 \tag{2.7}$$

Solving for P involves solving a matrix quadratic[2]. Once P is known, the second condition can be solved for Q:

$$(FQ+L)N + (FP+G)Q + M = 0 \tag{2.8}$$

The third condition can be solved for U to give:

$$U = -[F(I+P)+G]^{-1}[T + (FQ+L)(N-I)(Z_0-\bar{Z})] \tag{2.9}$$

[2]see Uhlig (1999) and Christiano (2002) for details

The matrices F, G, H, L, M, P & Q can be obtained analytically or numerically. By evaluating the system of equations at the point about which we are linearizing, we can obtain values for $T = (X_0, X_0, X_0, Z_0, Z_0)$.

Linearizing about any arbitrary value of the state variables is computationally no more complicated than the standard procedure of linearizing about the steady state. We merely add an extra term to the policy function, U, which can be easily computed using standard matrix algebra

We now consider linearization about two specific values. The first is the standard technique of linearizing about the steady state, $\bar{\theta} = \{\bar{X}, \bar{Z}\}$. Linearizing about the steady state sets $U = 0$, since the terms T and $(Z_0 - \bar{Z})$ in equation (2.9) are both zero. This is the standard technique laid out in Uhlig (1999) and Christiano (2002). We refer to this method as the "steady-state linearization")SSL) method.

The second special case is to linearize about the current state, $\theta_t = \{X_{t-1}, Z_t\}$. In this case, the conditions from equations eqref2.7 through eqref2.9 hold as written, but the policy function becomes $\tilde{X}_t = U$, since $\tilde{X}_{t-1} = \tilde{Z}_t = 0$. An exact solution for the constant, U, comes from equation eqref2.9 as above. Since the key dynamic equations are approximated about a different point each period, the values of the derivatives F, G, H, L & M and the constant T will change each period also. This means that values for P & Q will need to be found each period as well. However, P & Q are not directly used to find the new value of $\tilde{X}_t$, only to find the correct value of U. We refer to this method as the "current-state linearization" (CSL) method.

3 Approximation about the Current State

Our proposed method is to solve for a unique P, Q & U each period. We linearize about the current state , solve for P, Q & U and then use these values to find next periods state. We repeat to generate a whole stochastic time path. This means that we must use a modified version of (2.5) where the values of P, Q & U vary over time periods.

$$\tilde{X}_t = U_t + P_t \tilde{X}_{t-1} + Q_t \tilde{Z}_t \tag{3.1}$$

The coefficient values in (2.2) will also be time dependent. Iteratively substiuting versions of (3.1) into (2.2) yields the following equivalent of (2.6).

$$[(F_tP_{t+1}+G_t)P_t+H_t]\tilde{X}_{t-1}+[(F_tQ_{t+1}+L_t)N+(F_tP_{t+1}+G_t)Q_t+M_t]\tilde{Z}_t \\ +T_t+[F_tU_{t+1}+F_tP_{t+1}U_t)+G_tU_t+(F_tQ_{t+1}+L_t)(N-I)(Z_t-\bar{Z})=0 \quad (3.2)$$

There is no way to solve for P_t, Q_t & U_t each period without knowing their values in period $t+1$ first. The conditions which define these values – the equilvalents of (2.6) - (2.8) – are:

$$(F_tP_{t+1}+G_t)P_t+H_t=0 \quad (3.3)$$

$$(F_tQ_{t+1}+L_t)N+(F_tP_{t+1}+G_t)Q_t+M_t=0 \quad (3.4)$$

$$T_t+[F_tU_{t+1}+F_tP_{t+1}U_t)+G_tU_t+(F_tQ_{t+1}+L_t)(N-I)(Z_t-\bar{Z})=0 \quad (3.5)$$

One method for dealing with this problem is to approximate about a specific time path. With this method we must have a model which converges in the long-run to a steady state. If we assume this convergence occurs by period T we can solve for P_T, Q_T & U_T using the same equations as when approximating about the steady state since in period T the economy is assume to be at this steady state. From here we can use equations (3.3) - (3.5) to iteratively solve for earlier-period values of P, Q & U. The most obvious time path to choose is the one with no stochastic shocks beyond the current period as this is the most likely path to actually occur. However, the actual time path for the simulation will almost certainly differ from this path. If it deviates enough in a given period the approximation will be poor, just as in the steady state approximation case. This method, which we dub the "zero-shock path" (ZSP) approximation requires finding a unique value of P_t, Q_t & U_t for each period in the simulation. However once this is done, it is not necessary to recalculate these values for different Monte Carlos.

As an alternative to the ZSP method, our CSL methods assumes the values of P, Q & U in periods t & $t+1$ are approximately the same. This assumption allows us to solve for each periods coefficients in isolation, without having to refer to next periods actual values. It has the advantage of not needing to solve for a benchmark time path via some other method. Indeed, it is not even necessary to solve for the steady state. Instead, we generate the time path as we solve and simulate each period. A disadvantage is that we must recalculate the unique values of P, Q & U each period in each Monte Carlo simulation.

In addition, our CSL method will have zero distance approximation error because the linearization point is todays state exactly. It will have additional approximation error relative to SSL and ZSP, however, due to the fact that it assumes the wrong values for future coefficients when solving for todays (i.e. using a quadratic solution every period). We refer to this error henceforth as the isolation approximation error.

The fundamental question of this paper is whether the isolation error associated with CSL is greater or less than the distance error associated with the SSL and ZSP methods.

4 Comparing Solution Methods Across Models

In this section, we compare the CSL solution method to other standard DSGE solution methods. These other methods include the analytical solution where possible, value function iteration, first order Taylor series approximation around the certainty equivalent steady-state, and second order Taylor series approximation around the certainty equivalent steady-state. We test the performance of these methods on five models.

For each method in each model, we compute the implied path of the aggregate capital stock in a standard impulse response function to a shock of one standard deviation beginning at the steady-state for 40 periods, in an impulse response function to a large shock in which the economy begins away from the steady-state for 40 periods, and to a simulation of the economy for 1,000 periods. We also compute the

Euler errors in each computed example and report the root mean squared value and the maximum absolute value for each solution method. The general specification of the Euler error for individual s in period t is in percent deviation from the theoretical target of $\eta = 1$.

$$\eta_{s,t} = \frac{\beta E\left[(1 + r_{t+1} - \delta)u'(c_{s+1,t+1})\right]}{u'(c_{s,t})} - 1 \quad \forall s, t \tag{4.1}$$

One drawback to testing different approximations of the policy function solutions to models characterized by systems of Euler equations using summary statistics of Euler errors of the form (4.1) is that the Euler error contains two sources of error. The first and most explicit source of error comes from the fact that the solution methods deliver approximations of the true policy function, and thereby create nonzero Euler errors. However, a second source of error enters into (4.1) because the approximated policy function must be used in approximating the integral in the numerator. As will be shown in the comparisons of solution methods in Section 4.1 in the model for which the analytical solution is known, the ranking of solution method accuracy can be distorted by these two sources of error. Although Euler errors are an imperfect measure of approximation method accuracy, they are a desirable second best when analytical solutions are not available as a benchmark.

The following subsections describe the five models we use to test the performance of the solution methods along with the results from the comparison of methods.[3]

4.1 Growth model with analytical solution

The following model was first studied by Brock and Mirman (1972) and is important because it characterizes a standard representative agent infinite horizon DSGE growth model that has a closed form solution. The firm produces output Y_t using aggregate capital K_t and aggregate labor L_t according to a standard Cobb-Douglas production function,

$$Y_t = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} \tag{4.2}$$

[3] More detailed descriptions of each of the models can be found in the Technical Appendix, which is available upon request.

where $\alpha \in (0,1)$ is the capital share of income and e^{z_t} represents total factor productivity. The law of motion for z_t is an AR(1) process.

$$z_{t+1} = \rho z_t + (1-\rho)\mu + \varepsilon_{t+1} \quad \text{where} \quad \varepsilon_t \sim N(0, \sigma^2) \tag{4.3}$$

The standard equations for the real wage w_t and real rental rate r_t are given by the standard first order conditions of the profit function with respect to capital and labor demand.

$$w_t = (1-\alpha) e^{z_t} \left(\frac{K_t}{L_t}\right)^{\alpha} \tag{4.4}$$

$$r_t = \alpha e^{z_t} \left(\frac{L_t}{K_t}\right)^{1-\alpha} \tag{4.5}$$

The representative household supplies labor $l_t = 1$ inelastically and chooses consumption c_t and savings k_{t+1} each period to maximize expected lifetime utility subject to a budget constraint.

$$V(k_t, z_t) = \max_{k_{t+1}} \ln c_t + \beta E_t \left[V(k_{t+1}, z_{t+1})\right] \tag{4.6}$$

$$\text{s.t.} \quad c_t + k_{t+1} = w_t + r_t k_t \tag{4.7}$$

Note that the budget constraint (4.7) implies that the capital fully depreciates each period $\delta = 1$.[4] This is the key assumption allowing for an analytical solution for the equilibrium policy function. Household optimization is characterized by a sequence of Euler equations.

$$\frac{1}{c_t} = \beta E\left[\frac{r_{t+1}}{c_{t+1}}\right] \quad \forall t \tag{4.8}$$

[4] The more general budget constraint in which capital only partially depreciates at rate $\delta \in (0,1)$ would be $c_t + k_{t+1} = w_t l_t + (1 + r_t - \delta) k_t$.

The capital, labor, and goods markets must clear in equilibrium.

$$L_t = 1 \tag{4.9}$$

$$K_t = k_t \tag{4.10}$$

$$C_t + K_{t+1} = Y_t \tag{4.11}$$

Using a standard definition of equilibrium and Howard's improvement algorithm, Sargent (1987, pp. 47-48) shows that the closed-form solution to this model is a policy function of the form,

$$k' = \psi(k, z) = \alpha\beta e^z k^\alpha. \tag{4.12}$$

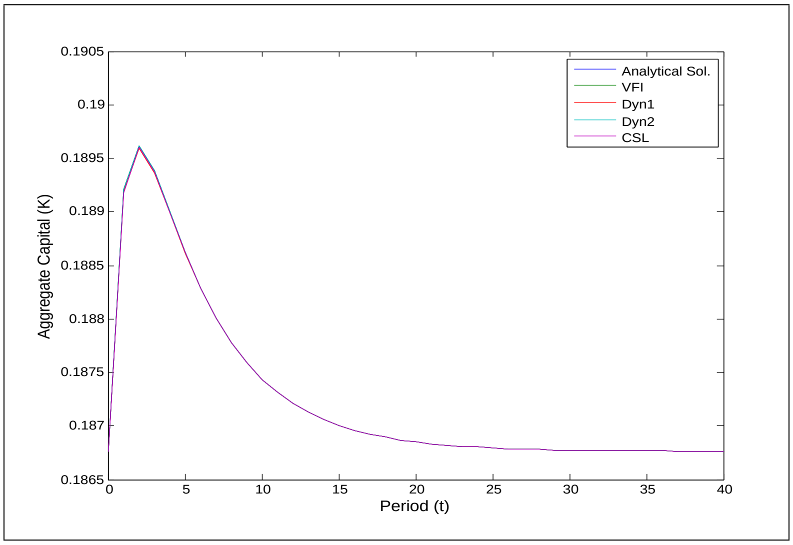
Table 1: Comparison of solution methods for growth model with analytical solution

		Analytical solution	Value function iteration	1st order approx. (Dynare)	2nd Order approx. (Dynare)	Current state linearization
IRF 1	RMSEE	$2.070e-16$	$8.052e-07$	$4.137e-05$	0.19410	$9.278e-05$
$(\bar{k}, \mu+\sigma)$	MAEE	$4.441e-16$	$4.362e-06$	$4.481e-05$	0.19413	$1.417e-04$
	RMSD	0	$1.156e-07$	$5.868e-06$	$3.577e-08$	$5.441e-06$
IRF 2	RMSEE	$2.781e-16$	$1.465e-06$	$6,438e-05$	0.1942	$2.854e-04$
$(1.1\bar{k}, \mu+2\sigma)$	MAEE	$6.661e-16$	$7.411e-06$	$2.348e-04$	0.1950	0.0017
	RMSD	0	$2.292e-07$	$1.749e-05$	$1.624e-06$	$4.799e-05$
Simulation	RMSEE	$2.513e-16$	$1.068e-05$	$8.240e-04$	$2.053e-05$	$4.479e-04$
(1,000 per.)	MAEE	$8.882e-16$	$2.270e-04$	0.0084	$3.002e-04$	0.0036
	RMSD	0	$9.554e-06$	$9.607e-04$	$2.599e-05$	$4.022e-04$
Time	seconds	0.26	2633.56	6.52	3.40	19.19

RMSEE is the root mean squared Euler error and MAEE is the maximum absolute Euler error from the particular simulation. RMSD is the root mean squared deviation from the analytical solution.

We implement a standard calibration of this model of $\alpha = 0.35$, $\beta = 0.96$, $\rho = 0.815$, $\mu = 0$, and $\sigma = 0.013$, and we solve the model using each of the five methods listed in Table 1. Figure 1 shows the five impulse response functions for the five solution methods for a one standard deviation shock to the model starting from the steady-state $(\bar{k}, \mu+\sigma)$. Figure 2 shows the five impulse response functions for the five solution methods for a two standard deviation shock to the model starting above the steady state $(1.1\bar{k}, \mu+2\sigma)$.

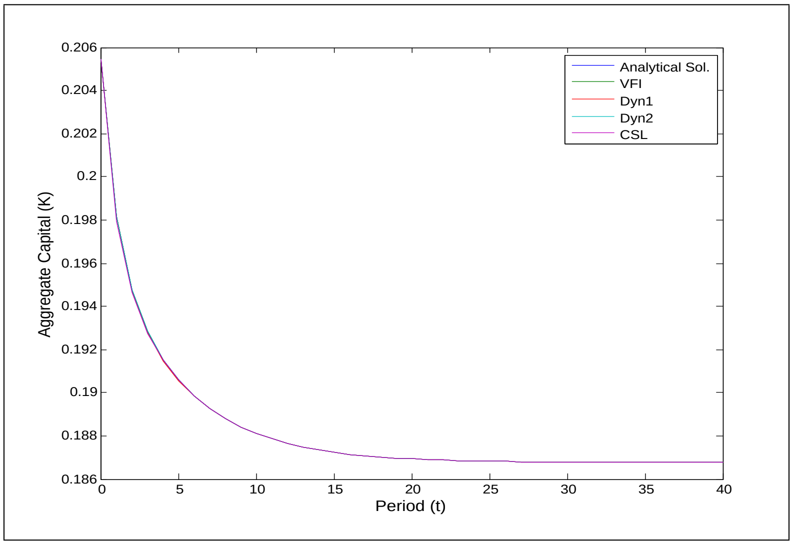
Figure 1: Impulse response functions from steady-state with one standard deviation shock for 5 solution methods: Analytical Solution Growth Model



Because we know the analytical solution to this model, we can compare our four approximation solution methods to the analytical solution in all three simulation experiments. CSL is closer to the know analytical solution for every method except the second order approximation in the first impulse response function and in the simulation. However, CSL drops to third place in accuracy in the second impulse response function with a bigger shock and deviation from the steady state, behind both the first and second order approximations.

In summary, Table 1 and Figures 1 and 2 show that CSL performs well as an approximating solution method for this growth model for which the closed form solution is known. In the analyses that follow for the other models, we will not be able to use proximity to the analytical solution because the analytical solution is not known. Instead, we will have to rely on Euler errors. One problem with the Euler errors is that the expectation of the marginal utility of consumption tomorrow includes a k_{t+2} term. To compute this integral, we have to use the policy function for the given model. This means that our Euler errors have two sources of error. They have approximation error resulting from the particular solution method directly as well as approximation error of the integral in the expectation because of the inclusion

Figure 2: Impulse response functions away from steady-state with two standard deviation shock for 5 solution methods: Analytical Solution Growth Model



of the approximated policy function in the integral. Although we note this potential drawback to using Euler errors as our measure for ranking accuracy, the degree of the problem is likely small. Further, we know of no suitable alternative.

4.2 Standard infinite horizon growth model

For our standard infinite horizon growth model, we simply take the Brock and Mirman model from Section 4.1 and allow capital to not fully depreciate each period $\delta \in [0, 1]$ and let the period utility function be the more general CRRA form rather than log utility. Aggregate production is characterized by the same Cobb-Douglas production function as in (4.2) with the same law of motion for productivity (4.3) and the same first order conditions characterizing the real wage w_t (4.4) and the real rental rate r_t (4.5).

Again, households supply labor inelastically $l_t = 1$. The value function now has the general CRRA period utility function $u(c_t)$, and the household budget constraint now includes a depreciation cost of capital δ.

$$V(k_t, z_t) = \max_{k_{t+1}} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \beta E_t \left[V(k_{t+1}, z_{t+1})\right] \tag{4.13}$$

$$\text{s.t.} \quad c_t + k_{t+1} = w_t + (1 + r_t - \delta)k_t \tag{4.14}$$

Household optimization is characterized by a sequence of Euler equations.

$$u'(c_t) = \beta E\Big[(1 + r_{t+1} - \delta)\, u'(c_{t+1})\Big] \quad \forall t \tag{4.15}$$

The capital, labor, and goods markets must clear in equilibrium.

$$L_t = 1 \tag{4.16}$$

$$K_t = k_t \tag{4.17}$$

$$C_t + K_{t+1} - (1-\delta)K_t = Y_t \tag{4.18}$$

Using a standard rational expectations functional equilibrium definition, it can be shown that the solution to this model is a stationary household policy function of the state $k' = \psi(k, z)$ that represents a fixed point in the equilibrium value function $V(k, z)$ in the following equilibrium Bellman equation.

$$V(k, z) = \max_{k'=\psi(k,z)} \frac{\Big[w(k, z) + \big(1 + r(k, z) - \delta\big)k - k'\Big]^{1-\gamma} - 1}{1-\gamma} + \beta E_{z'|z}\Big[V\big(k', z'\big)\Big] \tag{4.19}$$

$$k' = \psi(k, z) \tag{4.20}$$

We implement a standard calibration of this model of $\alpha = 0.35$, $\beta = 0.96$, $\delta = 0.05$, $\gamma = 2.5$, $\rho = 0.815$, $\mu = 0$, $\sigma = 0.013$, in which the common variables to the model in Section 4.1 have the same values. We solve the model using each of the four methods listed in Table 2. In contrast to Table 1, notice that Table 2 does not have a row for root mean squared deviation (RMSD) from the analytical solution. Because we cannot solve for the analytical solution for this more general infinite horizon model,

Table 2: Comparison of solution methods for standard infinite horizon growth model

		Value function iteration	1st order approx. (Dynare)	2nd Order approx. (Dynare)	Current state linearization
IRF 1	RMSEE	$2.868e-06$	$1.933e-04$	0.9876	$1.078e-05$
$(\bar{k}, \mu+\sigma)$	MAEE	$8.948e-06$	$2.119e-04$	0.9876	$1.787e-05$
	RMSD	0	$2.954e-03$	$4.820e-05$	$2.889e-03$
IRF 2	RMSEE	$3.742e-06$	$2.356e-04$	0.9822	$7.254e-05$
$(1.1\bar{k}, \mu+2\sigma)$	MAEE	$1.415e-05$	$7.589e-04$	0.9889	$1.568e-04$
	RMSD	0	$3.707e-03$	$8.473e-05$	$7.011e-03$
Simulation	RMSEE	$6.888e-04$	$4.417e-04$	$1.965e-04$	$9.578e-04$
(1,000 per.)	MAEE	$2.022e-02$	$5.225e-03$	$5.611e-03$	$2.150e-02$
	RMSD	0	0.4069	0.3398	0.4014
Time	seconds	21576.87	4.22	3.86	14.55

RMSEE is the root mean squared Euler error and MAEE is the maximum absolute Euler error from the particular simulation. RMSD is the root mean squared deviation from the value function iteration solution.

we must use Euler errors exclusively for measuring the accuracy of each method. However, as noted in the introduction to this section 4 and as illustrated in the rankings of Table 1, Euler errors are not a perfect measure of accuracy.

Figure 1 shows the four impulse response functions for the four solution methods for a one standard deviation shock to the model starting from the steady-state $(\bar{k}, \mu+\sigma)$. Figure 2 shows the four impulse response functions for the four solution methods for a two standard deviation shock to the model starting above the steady state $(1.1\bar{k}, \mu+2\sigma)$.

Figure 3: Impulse response functions from steady-state with one standard deviation shock for 4 solution methods: Standard Growth Model

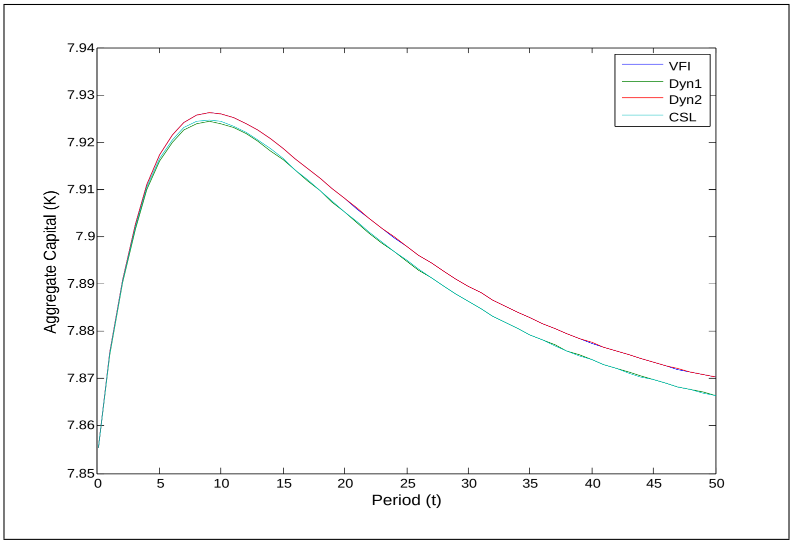
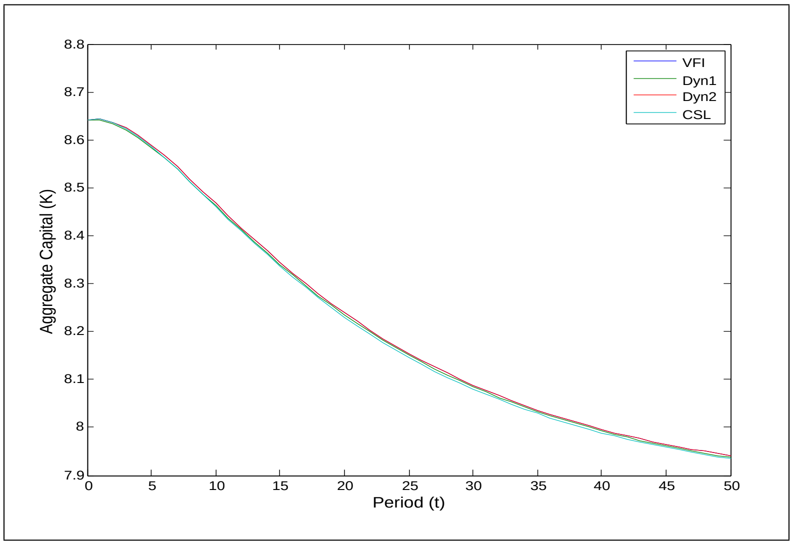


Figure 4: Impulse response functions away from steady-state with two standard deviation shock for 4 solution methods: Standard Growth Model



4.3 Growth model with progressive marginal tax rate

In this section, we augment the model from section 4.2 to include taxes and transfers. In particular, we include a marginal tax rate on labor and capital income faced by the household. The significance in adding this tax function is that the incentives faced by the household when the state is far away from the steady state are very different from the incentives around the steady-state. The household's Bellman equation is the following,

$$V(k_t, z_t) = \max_{k_{t+1}} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \beta E_t\{V(k_{t+1}, z_{t+1})\}$$

$$c_t = w_t + (1 + r_t - \delta)k_t - \tau_t[w_t + (r_t - \delta)k_t] - k_{t+1} + d_t \tag{4.21}$$

where τ_t is the progressive marginal tax rate and d_t is a lump-sum transfer. The tax rate has a t-subscript because it changes as a function of the household's labor and capital income.

$$\tau_t = \tau\Big(w_t + [r_t - \delta]k_t\Big) \tag{4.22}$$

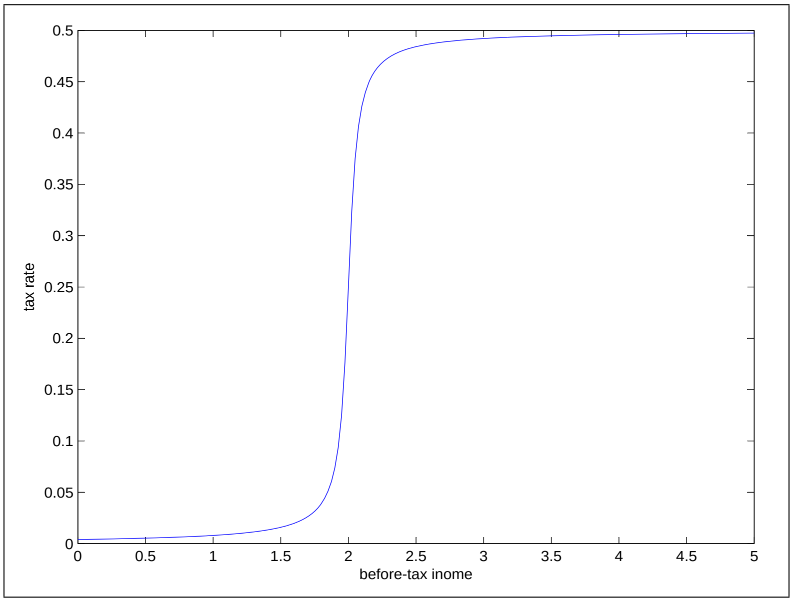
For our marginal tax rate function, we choose the following continuous functional form,

$$\begin{gathered}\tau_t = \tau_1 + \left[\frac{1}{\pi}\arctan\big\{a\,x_t + b\big\} + \frac{1}{2}\right](\tau_2 - \tau_1) \\ \text{where} \quad x_t \equiv w_t + (r_t - \delta)k_t\end{gathered} \tag{4.23}$$

for which $\lim\limits_{x\to\infty} \tau_t = \tau_2 \in (0,1)$, $\lim\limits_{x\to-\infty} \tau_t = \tau_1 < \tau_2$, and the marginal tax rate is everywhere increasing in income $\tau'(x) > 0$. The parameter $a > 0$ governs the sharpness of the transition from τ_1 to τ_2, and the parameter b shifts the function from right to left. Figure 5 shows the marginal tax rate as a function of gross taxable income for $\tau_1 = 0.10$, $\tau_2 = 0.23$, $a = 4$, and $b = -12$.

Because this is a representative agent model, the intuition for the marginal tax rate function τ_t in (4.23) is different from the standard intuition in a model with heterogeneous agents. Because this model only has one agent, variation in income happens only over time. So this tax function implies that the government raises

Figure 5: Functional form for varying marginal tax rates: $\tau_1 = 0.10$, $\tau_2 = 0.23$, $a = 4$, $b = -12$



the income tax rate in good times and lowers it in bad times. We have chosen the calibration of the function shown in Figure 5 so that the certainty-equivalent steady state is at a low marginal tax rate. In this case, the incentives faced by the household are much more distorted in good times than they are in bad times.

We assume a balanced government budget constraint each period given by the following equation.

$$d_t = \tau_t[w_t + (r_t - \delta)k_t] \tag{4.24}$$

This tax policy gives the following household Euler equation which along with (4.3) governs the model's dynamics,

$$\begin{aligned} c_t^{-\gamma} &= \beta E_t\left[c_{t+1}^{-\gamma}\left(1 + (1-\tau_{t+1})(r_{t+1}-\delta) - \frac{\partial \tau_{t+1}}{\partial k_{t+1}}\left[w_{t+1} + (r_{t+1}-\delta)k_{t+1}\right]\right)\right] \\ \text{where} \quad \frac{\partial \tau_{t+1}}{\partial k_{t+1}} &= \left(\frac{\tau_2 - \tau_1}{\pi}\right)\left(\frac{a(r_{t+1}-\delta)}{1 + \left[a\big(w_{t+1} + [r_{t+1}-\delta]k_{t+1}\big) + b\right]^2}\right) \end{aligned} \tag{4.25}$$

where c_t is given by the budget constraint in (4.21).

Table 3: Comparison of solution methods for growth model with varying marginal tax rates

		Policy function iteration	1st order approx. (Dynare)	2nd Order approx. (Dynare)	Current state linearization
IRF 1	RMSEE	$5.245e-03$	$5.409e-04$	.9855	$2.827e-05$
$(\bar{k}, \mu+\sigma)$	MAEE	$5.390e-03$	$5.669e-04$	.9856	$1,254e-04$
	RMSD	0	$2.976e-03$	$1.641e-05$	$2.736e-03$
IRF 2	RMSEE	$5.010e-03$	$5.363e-04$	.9862	$9.984e-05$
$(1.1\bar{k}, \mu+2\sigma)$	MAEE	$5.189e-03$	$6.015e-04$	.9871	$2.719e-04$
	RMSD	0	$3.847e-03$	$9.383e-05$	$5.000e-03$
Simulation	RMSEE	$5.341e-03$	$9.072e-04$	$5.927e-04$	$7.508e-03$
(1,000 per.)	MAEE	$1.824e-02$	$1.824e-02$	$1.201e-02$	$8.252e-03$
	RMSD	0	$1.711e-03$	$3.137e-04$	$6.010e-03$
Time	seconds		5.44	4.12	14.87

RMSEE is the root mean squared Euler error and MAEE is the maximum absolute Euler error from the particular simulation. RMSD is the root mean squared deviation from the policy function iteration solution.

As with the model in Section 4.2, we implement a standard calibration of this model of $\alpha = 0.35$, $\beta = 0.96$, $\delta = 0.05$, $\gamma = 2.5$, $\rho = 0.815$, $\mu = 0$, $\sigma = 0.013$, in which the common variables to the model in Section 4.1 have the same values. We solve the model using each of the four methods listed in Table 3. As with Table 2, notice that Table 3 does not have a row for root mean squared deviation (RMSD) from the analytical solution. Because we cannot solve for the analytical solution for this more general infinite horizon model, we must use Euler errors exclusively for measuring the accuracy of each method. However, as noted in the introduction to this section 4 and as illustrated in the rankings of Table 1, Euler errors are not a perfect measure of accuracy.

Figure 1 shows the four impulse response functions for the four solution methods for a one standard deviation shock to the model starting from the steady-state $(\bar{k}, \mu + \sigma)$. Figure 2 shows the four impulse response functions for the four solution methods for a two standard deviation shock to the model starting above the steady state $(1.1\bar{k}, \mu + 2\sigma)$.

Figure 6: Impulse response functions from steady-state with one standard deviation shock for 4 solution methods: Varying Tax Rates Model

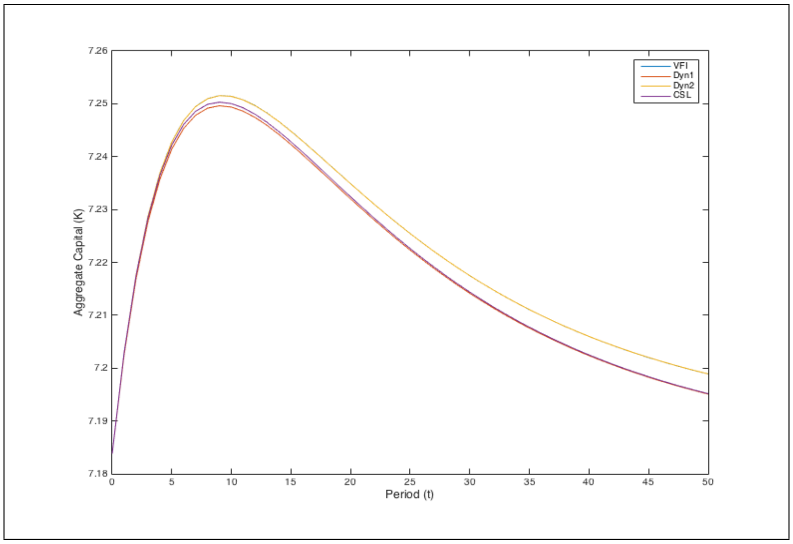
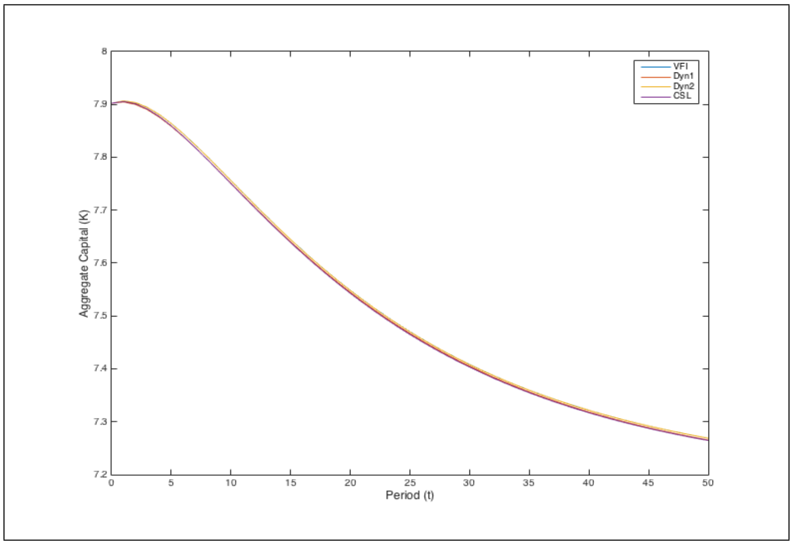


Figure 7: Impulse response functions away from steady-state with two standard deviation shock for 4 solution methods: Varying Tax Rates Model



4.4 Growth Model with Unstable Fiscal Policy

We modify the model from section 4.2 to include a payroll taxes and an independently determined set level of transfers. The household's dynamic program becomes:

$$V(k_t, z_t) = \max_{k_{t+1}} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \beta E_t\{V(k_{t+1}, z_{t+1})\}$$

$$c_t = (1-\tau)w_t + (1 + r_t - \delta)k_t - k_{t+1} + d \tag{4.26}$$

where τ_t is the time-varying tax rate and d is a lump-sum transfer that remains fixed over time.

The government's budget constraint each period is given by (4.27).

$$B_{t+1} = \tau_t w_t + (1 + r_t - \delta)B_t - d \tag{4.27}$$

And the government now chooses the tax rate based upon the level of its net stock of assets as show in equation (4.28)

$$\tau_t = \begin{cases} 0 & \text{if } B_t > B_{max} \\ \bar{\tau}\frac{B_{max} - B_t}{B_{max} - B_{upp}} & \text{if } B_{max} > B_t > B_{upp} \\ \bar{\tau} & \text{if } B_{upp} > B_t > B_{low} \\ \bar{\tau} + (1-\bar{\tau})\frac{B_{low} - B_t}{B_{low} - B_{min}} & \text{if } B_{low} > B_t > B_{min} \\ 1 & \text{if } B_{min} > B_t \end{cases} \tag{4.28}$$

This gives the following Euler equation which along with (4.3) governs the model's dynamics.

$$c_t^{-\gamma} = \beta E_t\{c_{t+1}^{-\gamma}(1 + r_{t+1} - \delta)\}; \tag{4.29}$$

$$c_t = (1-\tau_t)w_t + (1 + r_t - \delta)k_t - k_{t+1} + d$$

Aggregate capital available is the sum of private and public capital.

$$K_t = k_t + B_t \tag{4.30}$$

Points to stress.

- There are three steady states; two stable and one unstable. Cannot simulate using Dynare, since the model at the central steady state is unstable we violate the Blanchard-Kahn conditions.
- We can ignore and simulate anyway, however if we write our own code. Unfortunately, the quadratic approximation code does not exist.
- Value-function and Policy-function iterations also fail, for unknown reasons. In any case these take several hours for earlier models.
- So we have two methods that work, SSL and CSL. These we can compare.
- Note we cannot choose to linearize about the long-run SS if we do not know which one the model tends to.
- Even if we do, the transition dynamics appear highly innaccurate for SS approximation.

Figure 8: Imploding Fiscal Policy $B_0 = 0, \sigma = 0$

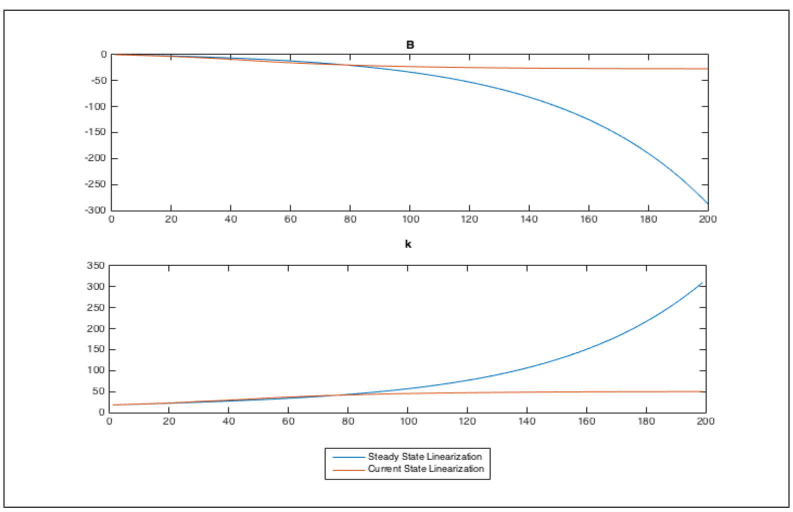


Figure 9: Exploding Fiscal Policy $B_0 = 2\bar{B}, \sigma = 0$

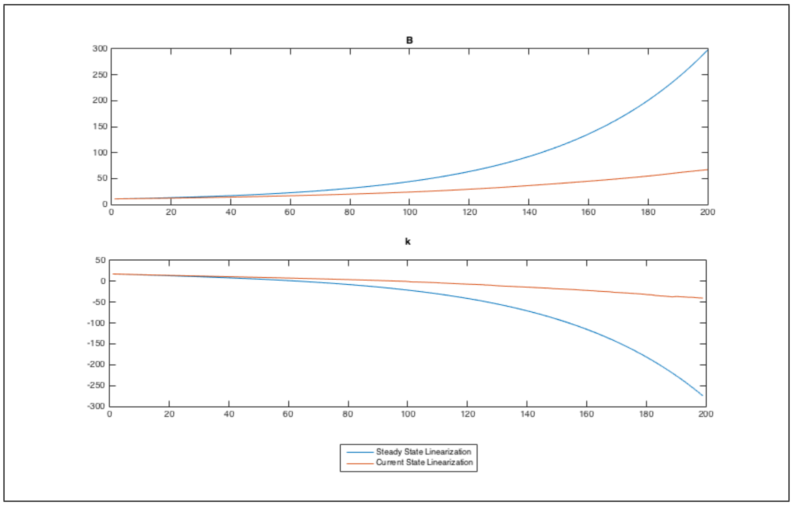
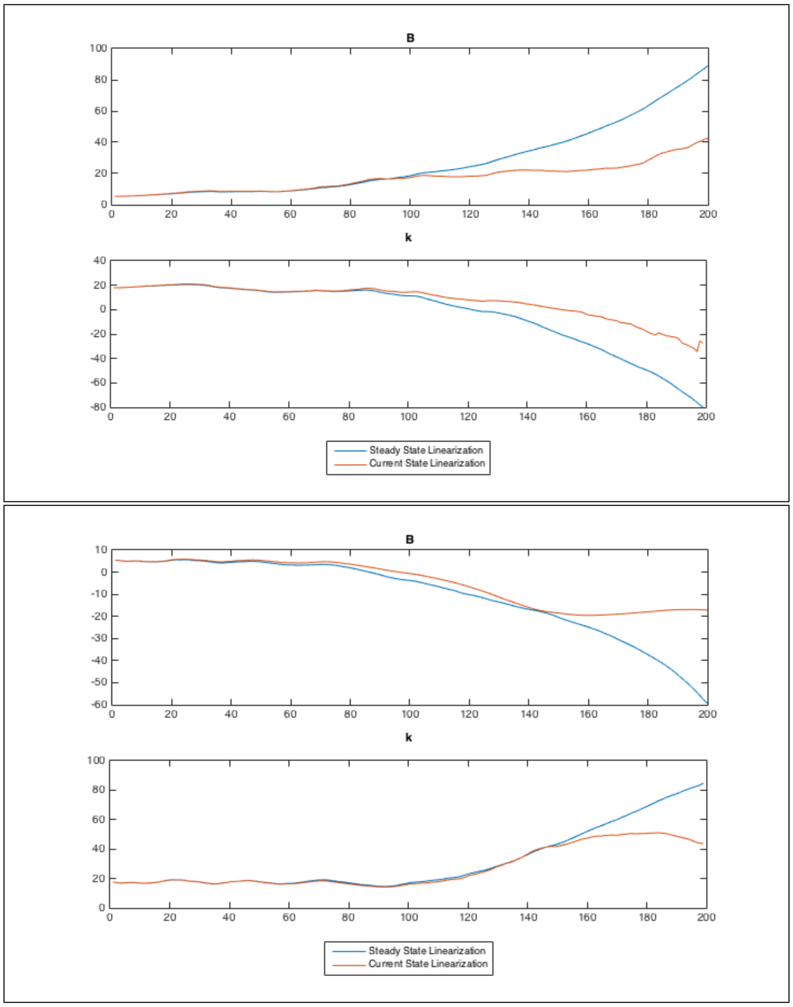


Figure 10: Sensitivity to Shocks $B_0 = 0, \sigma = 0$



5 Conclusion

CSL does not obviously dominate quadratic approximations for most stable models. Not inaccurate, but too time consuming.

CSL allow us to simulate unstable models very quickly, however. We can solve and simulate models with no steady state in a relatively short time. This allows us to begin using unblanced growth models, for example.

APPENDIX

A-1 Details of value function iteration

Heer and Maussner (2008, pp. 661-662) show that the Tauchen (1986) method of using a Markov approximation of the AR(1) process for productivity in (4.3) is more accurate than Gauss-Hermite continuous integration of the integral of next period's value function. They find that the extrapolation error beyond the end points in the discrete approximation of the support in Gauss-Hermite quadrature outweighs the approximation error in the more-discrete Tauchen method.

i. Choose a grid over the state (k,z) over which to evaluate the functions.

ii. Make an initial guess of the value function $V^i(k,z)$ over that grid.

   - A good initial guess is $V^0(k,z)=0$ for all k and z.

iii. Approximate the grid version of the value function $V^i(k,z)$ with a continous approximation of the value function $\tilde{V}^i(k,z)$.

   - The range of a utility function is $(-\infty,\infty)$, so a standard nth order polynomial is probably a fine approximating function.

iv. Plug continuously approximated guess $\tilde{V}^i(k',z')$ into the right-hand-side of (4.13) and solve for the policy function $\psi^i(k,z)$ that maximizes the Bellman equation at each grid point in the state space (k,z). This optimization will imply a new value function on the grid $V^{i+1}(k,z)$.

$$V^{i+1}(k,z)=\max_{\psi^i(k,z)} u\Big(w(k,z)+[1+r(k,z)-\delta]k-\psi^i(k,z)\Big)+\beta E_{z'|z}\tilde{V}^i\big(k',z'\big) \tag{A.1.1}$$

v. Let $\|\cdot\|$ be a norm on the space of grid value functions $V(k,z)$. If $\|V^{i+1}(k,z)-V^i(k,z)\|\leq\phi$, then we have found the fixed point. If $\|V^{i+1}(k,z)-V^i(k,z)\|>\phi$, the repeat steps (ii) through (v).

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TECHNICAL APPENDIX

T-1 Detailed Description of Growth Model with Analytical Solution

This section of the Technical Appendix provides a more detailed supplementary description of the model used in Section 4.1.

The following model was first studied by Brock and Mirman (1972) and is important because it characterizes a standard representative agent infinite horizon DSGE growth model that has a closed form solution. The firm produces output Y_t using aggregate capital K_t and aggregate labor L_t according to a standard Cobb-Douglas production function,

$$Y_t = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} \tag{4.2}$$

where $\alpha \in (0,1)$ is the capital share of income and e^{z_t} represents total factor productivity. The law of motion for z_t is an AR(1) process.

$$z_{t+1} = \rho z_t + (1-\rho)\mu + \varepsilon_{t+1} \quad \text{where} \quad \varepsilon_t \sim N(0,\sigma^2) \tag{4.3}$$

The firm's optimal capital and labor demands come from maximizing the real profit function,

$$Pr = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t \tag{T.1.1}$$

where w_t is the real wage and r_t is the real rental rate. The solutions for capital and labor demand are characterized by the standard first order conditions.

$$w_t = (1-\alpha) e^{z_t} \left(\frac{K_t}{L_t}\right)^{\alpha} \tag{4.4}$$

$$r_t = \alpha e^{z_t} \left(\frac{L_t}{K_t}\right)^{1-\alpha} \tag{4.5}$$

The representative household supplies labor $l_t = 1$ inelastically and chooses consumption c_t and savings k_{t+1} each period to maximize expected lifetime utility subject to a budget constraint.

$$V(k_t, z_t) = \max_{k_{t+1}} \ln c_t + \beta E_t \left[V(k_{t+1}, z_{t+1})\right] \tag{4.6}$$

$$\text{s.t.} \quad c_t + k_{t+1} = w_t + r_t k_t \tag{4.7}$$

Note that the budget constraint (4.7) implies that the capital fully depreciates each period $\delta = 1$.[5] This is the key assumption allowing for an analytical solution for the equilibrium policy function. Household optimization is characterized by a sequence of Euler equations.

$$\frac{1}{c_t} = \beta E\left[\frac{r_{t+1}}{c_{t+1}}\right] \quad \forall t \tag{4.8}$$

[5] The more general budget constraint in which capital only partially depreciates at rate $\delta \in (0,1)$ would be $c_t + k_{t+1} = w_t l_t + (1 + r_t - \delta) k_t$.

The capital, labor, and goods markets must clear in equilibrium.

$$L_t = 1 \tag{4.16}$$
$$K_t = k_t \tag{4.17}$$
$$C_t + K_{t+1} = Y_t \tag{4.11}$$

By Walras' Law, one of the market clearing conditions is redundant. In our computation and definition of the equilibrium, we will ignore the goods market clearing constraint (4.11).

Definition T.1.1 (Rational expectations functional equilibrium for closed–form solution growth model). Let the state in period t be (k, z). A rational expectations functional equilibrium is a capital investment function $k' = \psi(k, z|\Omega)$ and value function $V(k, z)$ and price functions $w(k, z)$ and $r(k, z)$, and beliefs Ω such that:

i. households optimize according to (4.8),

ii. firms optimize according to (4.4) and(4.5),

iii. markets clear according to (4.16) and (4.17),

iv. and beliefs about the distribution over uncertainty are correct.

Substituting the market clearing conditions (4.16) and (4.17) into the firm's first order conditions (4.4) and(4.5) give the equilibrium expressions for the price functions.

$$w(k, z) = (1 - \alpha)e^z k^\alpha \tag{T.1.2}$$
$$r(k, z) = \alpha e^z k^{\alpha-1} \tag{T.1.3}$$

Sargent (1987, pp. 47-48) shows that the closed form solution to this problem can either be solved for by successively evaluating the Bellman equation in (4.6) starting from an initial value function guess or by guessing a policy function that takes the form of a fixed percent of income and iterating on the Euler equation (Howard policy improvement algorithm). Both of these methods give the following closed form equilibrium solution to the capital savings policy function. However, the Howard policy improvement method is much faster.

$$k' = \psi(k, z) = \alpha\beta e^z k^\alpha. \tag{4.12}$$

T-2 Detailed Description of Standard Infinite Horizon Growth Model

This section of the Technical Appendix provides a more detailed supplementary description of the model used in Section 4.2.

For our standard infinite horizon growth model, we simply take the Brock and Mirman model from Section 4.1 and allow capital to not fully depreciate each period $\delta \in [0,1]$ and let the period utility function be the more general CRRA form rather than log utility. The firm produces output Y_t using aggregate capital K_t and aggregate labor L_t according to a standard Cobb-Douglas production function,

$$Y_t = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} \tag{4.2}$$

where $\alpha \in (0,1)$ is the capital share of income and e^{z_t} represents total factor productivity. The law of motion for z_t is an AR(1) process.

$$z_{t+1} = \rho z_t + (1-\rho)\mu + \varepsilon_{t+1} \quad \text{where} \quad \varepsilon_t \sim N(0,\sigma^2) \tag{4.3}$$

The firm's optimal capital and labor demands come from maximizing the real profit function,

$$Pr = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t \tag{T.1.1}$$

where w_t is the real wage and r_t is the real rental rate. The solutions for capital and labor demand are characterized by the standard first order conditions.

$$w_t = (1-\alpha)e^{z_t}\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{4.4}$$

$$r_t = \alpha e^{z_t}\left(\frac{L_t}{K_t}\right)^{1-\alpha} \tag{4.5}$$

The representative household supplies labor $l_t = 1$ inelastically and chooses consumption c_t and savings k_{t+1} each period to maximize expected lifetime utility subject to a budget constraint. The value function now has the general CRRA period utility function $u(c_t)$, and the household budget constraint now includes a depreciation cost of capital δ.

$$V(k_t, z_t) = \max_{k_{t+1}} \frac{c_t^{1-\gamma} - 1}{1-\gamma} + \beta E_t\left[V(k_{t+1}, z_{t+1})\right] \tag{4.13}$$

$$\text{s.t.} \quad c_t + k_{t+1} = w_t + (1 + r_t - \delta)k_t \tag{4.14}$$

Household optimization is characterized by the following sequence of Euler equations.

$$u'(c_t) = \beta E\Big[(1 + r_{t+1} - \delta)\, u'(c_{t+1})\Big] \quad \forall t \tag{4.15}$$

The solution to the household's problem is a stationary policy function for current-period savings for the next period k_{t+1} as a function of current-period capital holdings k_t, real wage w_t, and real rental rate r_t.

$$k_{t+1} = \psi(k_t, w_t, r_t) \quad \forall t \tag{T.2.1}$$

The capital, labor, and goods markets must clear in equilibrium.

$$L_t = 1 \tag{4.16}$$
$$K_t = k_t \tag{4.17}$$
$$C_t + K_{t+1} - (1 - \delta K_t) = Y_t \tag{4.18}$$

By Walra's Law, one of the market clearing conditions is redundant. In our computation and definition of the equilibrium, we will ignore the goods market clearing constraint (4.18).

Definition T.2.2 (Rational expectations functional equilibrium for standard infinite horizon growth model). Let the state in period t be (k,z). A rational expectations functional equilibrium is a capital investment function $k' = \psi(k,z|\Omega)$ and value function $V(k,z)$ and price functions $w(k,z)$ and $r(k,z)$, and beliefs Ω such that:

i. households optimize according to (4.15),

ii. firms optimize according to (4.4) and(4.5),

iii. markets clear according to (4.16) and (4.17),

iv. and beliefs about the distribution over uncertainty are correct.

Substituting the market clearing conditions (4.16) and (4.17) into the firm's first order conditions (4.4) and(4.5) give the equilibrium expressions for the price functions.

$$w(k,z) = (1-\alpha)e^z k^\alpha \tag{T.1.2}$$
$$r(k,z) = \alpha e^z k^{\alpha-1} \tag{T.1.3}$$

Plugging the functions (T.1.2) and (T.1.3) into the household's policy function in (T.2.1) gives the following equilibrium household policy function.

$$k' = \psi\big(k, w(k,z), r(k,z)\big) \quad \Rightarrow \quad k' = \psi(k,z) \tag{4.20}$$

The household value function (4.13) can now be written in equilibrium form.

$$V(k,z) = \max_{k'=\psi(k,z)} \frac{\Big[w(k,z) + \big(1 + r(k,z) - \delta\big)k - k'\Big]^{1-\gamma} - 1}{1-\gamma} + \beta E_{z'|z}\Big[V\big(k',z'\big)\Big] \tag{4.19}$$

The solution to this model and equilibrium definition is a stationary policy function $k' = \psi(k,z)$ (4.20) that represents a fixed point in the value function $V(k,z)$ in the equilibrium Bellman equation (4.19).

T-3 Problem with Euler error comparison across solution methods

Evaluation of Euler errors is an important test of whether a computational solution method has found the equilibrium. However, the Euler errors can only describe how closely the equilibrium conditions are satisfied conditional on the households in the model following the particular form of policy function. For models without an analytical solution, all computed policy functions are approximations. Therefore, the Euler errors for different approximation methods are not perfectly comparable.

To illustrate this principle, we look at the recursive equilibrium version of the household first order condition in (4.15) from the standard infinite horizon model in Section 4.2.

$$\begin{aligned}
&u'(c) = \beta E\big[(1+r'-\delta)\,u'(c')\big] \\
&\text{where} \quad c = w(k,z) + \big(1+r(k,z)-\delta\big)k - \psi(k,z) \\
&\text{and} \quad c' = w\big(\psi(k,z),z'\big) + \Big[1 + r\big(\psi(k,z),z'\big) - \delta\Big]\psi(k,z) - \psi\big(\psi(k,z),z'\big) \\
&\text{and} \quad k' = \psi(k,z) \quad \text{and} \quad k'' = \psi\big(\psi(k,z),z'\big)
\end{aligned} \tag{T.3.1}$$

An Euler error ε is simply a representation of the Euler equation written in the terms of the percent deviation of the discounted expected marginal utility of consumption tomorrow from the marginal utility of consumption today. This value should be zero in equilibrium.

$$\varepsilon \equiv \frac{\beta E\big[(1+r'-\delta)\,u'(c')\big]}{u'(c)} - 1 = 0 \tag{T.3.2}$$

Because the equilibrium policy function ψ appears in both the numerator and the denominator of the Euler error (T.3.1), comparison of Euler errors from different models are an imperfect tool for evaluating relative model accuracy.

The most accurate solution technique is to compare a simulated time path of the endogenous variables of an approximation to the known true time path of the endogenouse variables. This is the approach we take in Section 4.1 in which we study a model that has an analytical solution. These comparisons are illustrated in the third row of each section of Table 1. These rows show the root mean squared deviation of the local approximation method time paths from the analytical solution time path for three different simulations.

The other three models we study in Sections 4.2, 4.3, and 4.4 do not have analytical solutions. In this case, we compare the time paths of the local approximation methods to that of the global approximation method. These comparisons are illustrated in the third row of each section of Tables 2 and 3. These rows show the root mean squared deviation of the local approximation method time paths from the global approximation time path for three different simulations.

We still report different statistics on the Euler errors from the various simulated time paths in Tables 1, 2 and 3. But they should be interpreted with caution. In particular, the Euler errors for the second order perturbation approximation around the certainty equivalent steady state are always the largest even though the time paths from the second order perturbation approximations are always closer to the global approximation than the time paths of the first order perturbation approximations. This is because the squared terms multiply the size of any outlier errors.