A Computing the solution

In the next section, we give the formulas to sequentially compute higher order approximations. They imply a small set of tensor operations.

A.1 Perfect foresight case

The deterministic DSGE problem is written

$$f\left(V(x,u)\right)=0 \tag{1}$$

where $V\left(x,u\right)=\left[\begin{array}{c} g\left(g\left(x,u\right),0\right) \\ g\left(x,u\right) \\ x \\ u \end{array}\right]$

We assume the existence of $\bar{x}$ such that $f\left(V\left(\bar{x},0\right)\right)=0$. Our goal consists in recovering the derivatives of g knowing the derivatives of f evaluated at $V\left(\bar{x},0\right)$.

Derivatives of equation 1 can be written :

Order		
0		$f\left(V\right)=0$
1	x	$f^{'}.V_x^{'}$
	u	$f^{'}.V_u^{'}$
2	xx	$f^{''}.\left[V_x^{'},V_x^{'}\right]+f^{'}.\left[V_{xx}^{''}\right]$
	xu	$f^{''}.\left[V_x^{'},V_u^{'}\right]+f^{'}.\left[V_{xu}^{''}\right]$
	uu	$f^{''}.\left[V_u^{'},V_u^{'}\right]+f^{'}.\left[V_{uu}^{''}\right]$
3	xxx	$f^{'''}.\left[V_x^{'},V_x^{'},V_x^{'}\right]+3f^{''}.\left[V_{xx}^{''},V_x^{'}\right]+f^{'}.\left[V_{xxx}^{'''}\right]$
	xxu	$f^{'''}.\left[V_x^{'},V_x^{'},V_u^{'}\right]+2f^{''}.\left[V_x^{'},V_{xu}^{''}\right]+f^{''}.\left[V_{xx}^{''},V_u^{'}\right]+f^{'}.\left[V_{xxu}^{'''}\right]$
	xuu	$f^{'''}.\left[V_x^{'},V_u^{'},V_u^{'}\right]+2f^{''}.\left[V_{xu}^{''},V_u^{'}\right]+f^{''}.\left[V_x^{'},V_{uu}^{''}\right]+f^{'}.\left[V_{xuu}^{'''}\right]$
	uuu	$f^{'''}.\left[V_u^{'},V_u^{'},V_u^{'}\right]+3f^{''}.\left[V_{uu}^{''},V_u^{'}\right]+f^{'}.\left[V_{uuu}^{'''}\right]$

If we analyze this table carefully, we see that each line for any (k,l), the line can be written :

$$K_{x^k u^l}+f^{'}V_{x^k u^l}^{(k+l)} \tag{2}$$

where $K_{x^k u^l}$ is a function of lower order tensors. As a result, we can proceed sequentially and solve equation each equation $x^k u^l$ to get $V_{x^k u^l}^{(k+l)}$. However, equation 2 is not a simple linear equation as $V_{x^k u^l}^{(k+l)}$ has some internal structure.

To make it clear, let's compute the derivatives of V [1]:

[1] For order greater than 4, some complications arise from the fact indices order. For instance the dimensions of tensor $g_{xxxx}.\left[g_{xu},g_{xu}\right]$ should be reordered so as to represent an application $(x,x,u,u)\rightarrow g_{xxxx}.\left[g_{xu},g_{xu}\right]$.

Order		
0		$V(x,u) = \begin{bmatrix} g(g(x,u),0) \\ g(x,u) \\ x \\ u \end{bmatrix}$
1	x	$V_x^{'} = \begin{bmatrix} g_x g_x \\ g_x \\ I \\ 0 \end{bmatrix}$
	u	$V_u^{'} = \begin{bmatrix} g_x g_u \\ g_u \\ 0 \\ I \end{bmatrix}$
2	xx	$V_{xx}^{''} = \begin{bmatrix} g_{xx} \cdot [g_x, g_x] + g_x \cdot [g_{xx}] \\ g_{xx} \\ 0 \\ 0 \end{bmatrix}$
	xu	$V_{xu}^{''} = \begin{bmatrix} g_{xx} \cdot [g_x, g_u] + g_x \cdot [g_{xu}] \\ g_{xu} \\ 0 \\ 0 \end{bmatrix}$
	uu	$V_{uu}^{''} = \begin{bmatrix} g_{xx} \cdot [g_u, g_u] + g_x \cdot [g_{uu}] \\ g_{uu} \\ 0 \\ 0 \end{bmatrix}$
3	xxx	$V_{xxx}^{'''} = \begin{bmatrix} g_{xxx}^{'''} \cdot [g_x, g_x, g_x] + 3g_{xx} \cdot [g_{xx}, g_x] + g_x \cdot [g_{xxx}] \\ g_{xxx} \\ 0 \\ 0 \end{bmatrix}$
	xxu	$V_{xxu}^{'''} = \begin{bmatrix} g_{xxx}^{'''} \cdot [g_x, g_x, g_u] + 2g_{xx} \cdot [g_x, g_{xu}] + g_{xx} \cdot [g_{xx}, g_u] + g_x \cdot [g_{xxu}] \\ g_{xxu} \\ 0 \\ 0 \end{bmatrix}$
	xuu	$V_{xuu}^{'''} = \begin{bmatrix} g_{xxx}^{'''} \cdot [g_x, g_u, g_u] + 2g_{xx} \cdot [g_{xu}, g_u] + g_{xx} \cdot [g_x, g_{uu}] + g_x \cdot [g_{xuu}] \\ g_{xuu} \\ 0 \\ 0 \end{bmatrix}$
	uuu	$V_{uuu}^{'''} = \begin{bmatrix} g_{xxx}^{'''} \cdot [g_u, g_u, g_u] + 3g_{xx} \cdot [g_{uu}, g_u] + g_x \cdot [g_{uuu}] \\ g_{uuu} \\ 0 \\ 0 \end{bmatrix}$

By looking at this table, we see that $V_{x^k u^l}^{(k+l)}$ is determined by $g_{x^k u^l}^{(k+l)}$ provided that all previous derivatives of g have already been computed. When looking at the expressions more closely for orders greater than one, we see that we can write :

$$V_{x^k u^l}^{(k+l)} = \begin{bmatrix} L_{x^k u^l} + g_{x^k u^l}^{(k+l)} \cdot [g_x, ..., g_x, g_u, ..., g_u] + g_x \cdot \left[g_{x^k u^l}^{(k+l)} \right] \\ g_{x^k u^l}^{(k+l)} \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Combining this with the last result shows that $g_{x^k u^l}^{(k+l)}$ is the solution of :

$$K_{x^k u^l} + f' \begin{bmatrix} L_{x^k u^l} + g^{(k+l)}_{x^{k+l}} \cdot [g_x, ..., g_x, g_u, ..., g_u] + g_x \cdot \left[g^{(k+l)}_{x^k u^l} \right] \\ g^{(k+l)}_{x^k u^l} \\ 0 \\ 0 \end{bmatrix} = 0 \quad (4)$$

which is a sylvester equation if $l = 0$h

$$K_{x^k} + f'_+ L_{x^k} + (g_x + I)\, g^{(k)}_{x^k} + g^{(k)}_{x^k} \cdot [g_x, ..., g_x] = 0 \tag{5}$$

or a linear equation if $l > 0$

$$K_{x^k u^l} + f'_+ L_{x^k u^l} + + g^{(k)}_{x^k} \cdot [g_x, ..., g_x]\, (g_x + I)\, g^{(k)}_{x^k} = 0 \tag{6}$$

A.2 Perturbation w.r.t parameters

It is easy to compute the derivatives of the perfect foresight solution with respect to parameters, by writing the model as $f(V) = 0$ with

$$V = \begin{bmatrix} g(g(x,u,p), 0, p) \\ g(x,u,p) \\ x \\ u \\ p \end{bmatrix} \tag{7}$$

Some attention must be dedicated to the fact that rules for differentiating $g(g(x,u,p), 0, p)$ are not exactly the same as in the last case. They can still be easily automated.

A.3 Small expected risk expansion

Model is written $E[f(V(x,u,\sigma\epsilon))] = 0$ where ϵ is an i.i.d. shock and

$$V(x,u,\sigma) = \begin{bmatrix} g(g(x,u,\sigma), \sigma\epsilon, \sigma) \\ g(x,u,\sigma) \\ x \\ u \end{bmatrix} \tag{8}$$

We can prove that $V^{(k+l+m)}_{x^k u^l \sigma^m}$ can still be determined knowing derivatives with indices $\left(k', l', m'\right)$ such that $k' \leq k$, $l' \leq l$ and $m' < m$.

Yet one complication is arising here, $V^{(k+l+m)}_{x^k u^l \sigma^m}$ is now a stochastic variable (it depends on σ). Fortunately, we can still represent it as an explicit function of the derivatives of g, and implement corresponding operations.

We give them for order 3.

A.3.1 Derivatives of f w.r.t. σ

Order		$f(V)=0$
0	$\frac{\partial f}{\partial \sigma}$	$f^{'}.\left[V_{\sigma}^{'}\right]=0$
	$\frac{\partial^2 f}{\partial \sigma^2}$	$f^{''}.\left[V_{\sigma}^{'},V_{\sigma}^{'}\right]+f^{'}.\left[V_{\sigma\sigma}^{''}\right]=0$
1	$\frac{\partial f}{\partial \sigma}$	$f^{''}.\left[V_{x}^{'},V_{\sigma}^{'}\right]+f^{'}.\left[V_{x\sigma}^{''}\right]=0$
	$\frac{\partial^2 f}{\partial \sigma^2}$	$f^{'''}.\left[V_{x}^{'},V_{\sigma}^{'},V_{\sigma}^{'}\right]+2f^{''}.\left[V_{x\sigma}^{''},V_{\sigma}^{'}\right]+f^{''}.\left[V_{x}^{'},V_{\sigma\sigma}^{''}\right]+f^{'}.\left[V_{x\sigma\sigma}^{'''}\right]=0$

A.3.2 Derivatives of V w.r.t σ :

Order	$f(V)=0$
σ	$V_{\sigma}^{'}=\begin{bmatrix} g_x^{'}g_{\sigma}^{'}+g_{\sigma}^{'}+g_u^{'}\epsilon \\ g_{\sigma} \\ 0 \\ 0 \end{bmatrix}$
$\sigma\sigma$	$V_{\sigma\sigma}^{'}=\begin{bmatrix} g_{xx}^{''}\left[g_{\sigma}^{'},g_{\sigma}^{'}\right]+g_{xu}^{''}\left[g_{\sigma}^{'},\epsilon\right]+g_{x\sigma}^{''}\left[g_{\sigma}^{'}\right]+g_x^{'}g_{\sigma\sigma}^{''}+g_{x\sigma}^{''}+g_{\sigma\sigma}^{''}+g_{\sigma u}^{''}\epsilon+g_{ux}^{''}\left[\epsilon,g_{\sigma}^{'}\right]+g_{u\sigma}^{''}\epsilon+g_{uu}^{''}\left[\epsilon,\epsilon\right] \\ g_{\sigma\sigma}^{''} \\ 0 \\ 0 \end{bmatrix}$
$x\sigma$	$V_{x\sigma}^{''}=\begin{bmatrix} g_{xx}^{''}\left[g_x^{'},g_{\sigma}^{'}\right]+g_{x\sigma}^{''}\left[g_x^{'}\right]+g_{xu}^{''}\left[g_x^{'},\epsilon\right]+g_x^{'}g_{x\sigma}^{''} \\ g_{x\sigma}^{''} \\ 0 \\ 0 \end{bmatrix}$
$x\sigma\sigma$	$V_{x\sigma\sigma}^{''}=\begin{bmatrix} g_{xxx}^{''}\left[g_x^{'},g_{\sigma}^{'},g_{\sigma}^{'}\right]+g_{xx\sigma}^{''}\left[g_x^{'},g_{\sigma}^{'}\right]+g_{xxu}^{'''}\left[g_x^{'},g_{\sigma}^{'},\epsilon\right]+g_{xx}^{''}\left[g_{x\sigma}^{''},g_{\sigma}^{'}\right]+g_{xx}^{''}\left[g_x^{'},g_{\sigma\sigma}^{''}\right] \\ \ldots+g_{x\sigma x}^{'''}\left[g_x^{'},g_{\sigma}^{'}\right]+g_{x\sigma\sigma}^{'''}\left[g_x^{'}\right]+g_{x\sigma u}^{'''}\left[g_x^{'},\epsilon\right]+g_{x\sigma}^{''}\left[g_{x\sigma}^{''}\right] \\ \ldots+g_{xux}^{'''}\left[g_x^{'},\epsilon,g_{\sigma}^{'}\right]+g_{xu\sigma}^{'''}\left[g_x^{'},\epsilon\right]+g_{xuu}^{'''}\left[\epsilon,\epsilon\right]+g_{xu}^{''}\left[g_{x\sigma}^{''},\epsilon\right] \\ \ldots+g_{xx}^{''}\left[g_{x\sigma}^{''},g_{\sigma}^{'}\right]+g_{x\sigma}^{''}\left[g_{x\sigma}^{''}\right]+g_{xu}^{''}\left[g_{x\sigma}^{''},\epsilon\right]+g_x^{'}\left[g_{x\sigma\sigma}^{'''}\right] \\ g_{x\sigma\sigma}^{''} \\ 0 \\ 0 \end{bmatrix}$
$u\sigma$	$V_{u\sigma}^{'}=\begin{bmatrix} g_{xx}\left[g_u,g_{\sigma}\right]+g_{xu}\left[g_u,\epsilon\right]+g_{x\sigma}g_u+g_xg_{u\sigma} \\ g_{u\sigma} \\ 0 \\ 0 \end{bmatrix}$
$u\sigma\sigma$	$V_{u\sigma}^{'}=\begin{bmatrix} \text{(quite long expression)} \\ g_{u\sigma\sigma} \\ 0 \\ 0 \\ 0 \end{bmatrix}$

These expressions can be simplified dramatically by using the fact that $g_{\sigma}^{'}=0$, yielding :

Order	$f(V)=0$
0	$V_{\sigma}^{'} = \begin{bmatrix} g_{u}^{'}\epsilon \\ 0 \\ 0 \end{bmatrix}$ $V_{\sigma\sigma}^{''} = \begin{bmatrix} g_{x}^{'}g_{\sigma\sigma}^{''} + g_{\sigma\sigma}^{''} + g_{uu}^{''}\Sigma \\ g_{\sigma\sigma}^{''} \\ 0 \\ 0 \end{bmatrix}$
1	$V_{x\sigma}^{''} = \begin{bmatrix} g_{xx}^{''}\left[g_{x}^{'}, g_{\sigma}^{'}\right] + g_{x\sigma}^{''}\left[g_{x}^{'}\right] + g_{xu}^{''}\left[g_{x}^{'}, \epsilon\right] + g_{x}^{'}g_{x\sigma}^{''} \\ g_{x\sigma}^{''} \\ 0 \\ 0 \end{bmatrix}$ $V_{x\sigma\sigma}^{''} = \begin{bmatrix} +g_{x\sigma\sigma}^{'''}\left[g_{x}^{'}\right] + g_{xuu}^{'''}\left[g_{x}^{'}, \epsilon, \epsilon\right] + g_{xx}^{''}\left[g_{x}^{'}, g_{\sigma\sigma}^{''}\right] + g_{x}^{'}\left[g_{x\sigma\sigma}^{'''}\right] \\ g_{x\sigma\sigma}^{''} \\ 0 \\ 0 \end{bmatrix}$
	$V_{u\sigma}^{''} = \begin{bmatrix} g_{xx}\left[g_{u}, g_{\sigma}\right] + g_{xu}\left[g_{u}, \epsilon\right] + g_{x\sigma}g_{u} + g_{x}g_{u\sigma} \\ g_{u\sigma} \\ 0 \\ 0 \end{bmatrix}$ $V_{u\sigma\sigma}^{'''} = \begin{bmatrix} g_{xx}\left[g_{u}, g_{\sigma\sigma}\right] + g_{xuu}\left[g_{u}, \epsilon, \epsilon\right] + g_{x\sigma\sigma}\left[g_{u}\right] + g_{x}\left[g_{u\sigma\sigma}\right] \\ g_{u\sigma} \\ 0 \\ 0 \end{bmatrix}$

The equation we need to solve at each step is :

$$E_t\left\{K_{x^k u^l \sigma^m}\right\} + f^{'}E_t V^{(k+l+m)} = 0 \tag{9}$$

A.4 Small expected risk expansion with σ as a parameter

In this section, we assume that the parameter σ affects the model equations. The model is now written $f(V)$ where

$$V(x, u, \sigma\epsilon) = \begin{bmatrix} g\left(g\left(x, u, \sigma\right), \sigma\epsilon, \sigma\right) \\ g\left(x, u, \sigma\right) \\ x \\ u \\ \sigma \end{bmatrix} \tag{10}$$

The results are very similar to last section. Here we give only the relevant results.

A.4.1 Order 2

For $g_{\sigma}^{'}$ we solve:

$$\left(f_{+}^{'}\left(g_{x}^{'} + I\right) + f_{0}^{'}\right)g_{\sigma}^{'} = -f_{p}^{'} \tag{11}$$

For $g_{x\sigma}^{''}$ we solve :

$$f_{+}^{'}.g_{x\sigma}^{''}.\left[g_{x}^{'}\right] + \left(f_{+}^{'}g_{x}^{'} + f_{0}^{'}\right)g_{x\sigma}^{''} = -f^{''}.\left[V_{x}^{'}, \overline{V_{\sigma}^{'}}\right] - f_{+}^{'}\left(g_{xx}^{''}.\left[g_{x}^{'}, g_{\sigma}^{'}\right]\right) \tag{12}$$

For $g''_{\sigma\sigma}$ we solve :

$$\left(f'_+\left(g'_x+I\right)+f'_0\right)g''_{\sigma\sigma} = -f''_{pp}\left[\overline{V'_\sigma},\overline{V'_\sigma}\right]-f''_{++}\left[g'_u,g'_u\right].\Sigma \quad (13)$$

$$-f'_+\left(g''_{xx}.\left[g'_\sigma,g'_\sigma\right]+g''_{x\sigma}.\left[g'_\sigma\right]+g''_{x\sigma}+g''_{uu}.\Sigma\right) \quad (14)$$

With our choice of σ, we have $g'_\sigma=0$ and $g'_{x\sigma}=0$. The result is simply :

$$\left(f'_+\left(g'_x+I\right)+f'_0\right)g''_{\sigma\sigma} = -f''_{pp}-f''_{++}\left[g'_u,g'_u\right].\Sigma-f'_+\left(g''_{uu}.\Sigma\right) \quad (15)$$

A.4.2 Order 3

To be done