Description of Model B and exercises

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Abstract

Describes parameter specifications of model B and results to be reported.

1 Changes relative to the July 2007 notes

- It replaces the simulation method based on polynomial approximations with a cross-sectional distribution function using linear interpolation. This procedure is not that difficult either.
- It corrects a typo in the transition matrix. 0.009155 in the bottom left cell should have been 0.009115.
- Instead of using money at hand, decision rules for the case without aggregate uncertainty should be calculated using capital as the explanatory variable.
- A few little exercises where added and sometimes a bit more explanation about the required results is given.

2 Available data

All necessary data can be found at

http://www1.fee.uva.nl/toe/content/people/content/denhaan/datasuite.htm

- 1 stands for the aggregate bad state and 2 for the aggregate good state
- 1 stands for the unemployed and 2 for employed

3 Introduction

This note does the following:

- It gives the parameterization that came out of the 2004 SITE conference for model B. This set of parameters was also the basis of the comparisons that were done using the model without aggregate uncertainty immediately following this conference. Note that compared with the original proposal for the comparison, only two parameterizations are left, one with aggregate uncertainty and one without. Authors are free to consider *in addition* alternative specifications/models to highlight particular strengths and weaknesses of their models.

- It lists a set of model properties and accuracy tests. This part is not that different from the original proposal except for one aspect. It proposes to replace the stochastic simulation with a large finite number of agents with a simulation based on a continuum of agents, which is consistent with the model. This has several advantages. It eliminates cross-sectional sampling variation, which makes it easier to document some model properties, such as information about the wealth distribution, in a transparent and accurate manner. It also avoids the need to come up with a random number generator that (i) is portable across different platforms and software and (ii) can accurately generate truly large data sets. The procedure participants have to use is outlined below.

4 Model B

The economy is a production economy with aggregate shocks in which agents face different employment histories and partially insure themselves through (dis)saving in capital. For more details see Krusell and Smith (1998).

Problem for the individual agent. The economy consists of a unit mass of ex ante identical households. Each period, agents face an idiosyncratic shock ε that determines whether they are employed, $\varepsilon = 1$, or unemployed, $\varepsilon = 0$. An employed agent earns a wage rate of $\overline{w}_t$. An employed agent earns an after-tax wage rate of $(1-\tau_t)\overline{w}_t$ and an unemployed agent receives unemployment benefits $\mu\overline{w}_t$. Note that Krusell and Smith set μ equal to zero. This is the only difference with their model. Markets are incomplete and the only investment available is capital accumulation. The net rate of return on this investment is equal to $r_t - \delta$, where r_t is the rental rate and δ is the depreciation rate. Agent's i maximization problem is as follows:

$$\begin{aligned} &\max_{\{c_t^i, k_{t+1}^i\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} \\ \text{s.t.} \quad & c_t^i + k_{t+1}^i = r_t k_t^i + (1-\tau_t)\overline{w}_t \bar{l} \varepsilon_t^i + \mu \overline{w}_t (1-\varepsilon_t^i) + (1-\delta) k_t^i \\ & k_{t+1}^i \geq 0 \end{aligned} \tag{1}$$

Here c_t^i is the individual level of consumption, k_t^i is the agent's beginning-of-period capital, and $\bar{l}$ is the time endowment.

Firm problem. Markets are competitive and the production technology of the firm is characterized by a Cobb-Douglas production function. Consequently, firm heterogeneity is not an issue. Let K_t and L_t stand for per capita capital and the employment rate, respectively. Per capita output is given by

$$Y_t = a_t K_t^{\alpha} (\bar{l} L_t)^{1-\alpha} \tag{2}$$

and prices by

$$\overline{w}_t = (1-\alpha)\, a_t (\frac{K_t}{\bar{l} L_t})^{\alpha} \tag{3}$$

$$r_t = \alpha a_t (\frac{K_t}{\bar{l} L_t})^{\alpha - 1} \tag{4}$$

Aggregate productivity, a_t, is an exogenous stochastic process that can take on two values, $1 - \Delta^a$ and $1 + \Delta^a$.

Government The only role of the government is to tax employed agents and to redistribute funds to the unemployed. We assume that the government's budget is balanced each period. This implies that the tax rate is equal to

$$\tau_t = \frac{\mu u_t}{\bar{l} L_t}. \tag{5}$$

where $u_t = 1 - L_t$ denotes the unemployment rate in period t.

Exogenous driving processes. There are two stochastic driving processes. The first is aggregate productivity and the second is the employment status. Both are assumed to be first-order Markov processes. We let $\pi_{aa'\varepsilon\varepsilon'}$ stand for the probability that $a_{t+1} = a'$ and $\varepsilon_{t+1}^i = \varepsilon'$ when $a_t = a'$ and $\varepsilon_t^i = \varepsilon'$. These transition probabilities are chosen such that the unemployment rate can take on only two values. That is, $u_t = u^b$ when $a_t = a^b$ and $u_t = u^g$ when $a_t = a^g$ with $u^b > u^g$.

5 Parameter values

Two sets of parameter values are considered. In the first economy, there is no aggregate uncertainty. The transition probabilities for the idiosyncratic shock correspond to those of Krusell and Smith (1998) when the economy is always in the bad state. The aggregate capital stock is given and fixed, which results in a constant interest rate.

The parameter values of the second economy correspond with those of Krusell and Smith (1998) except that the unemployed receive unemployment benefits. Its parameter values are given in Tables 1 and 2. The discount rate, coefficient of relative risk aversion, share of capital in GDP, and the depreciation rate take on standard values. Unemployed people are assumed to earn a fixed fraction of 15% of the wage of the employed. The value of Δ^a is equal to 0.01 so that productivity in a boom, 1+Δ^a, is two percent above the value of productivity in a recession, 1-Δ^a. Business cycles are symmetric and the expected duration of staying in the same regime is eight quarters. The unemployment rate in a boom, u^g, is equal to 4% and the unemployment rate in a recession, u^b, is equal to 10%. The time endowment, $\bar{l}$, is chosen to normalize total labor supply in the recession to one. The average unemployment duration is 2.5 quarters conditional on staying in a recession and equal to 1.5 quarters conditional on staying in a boom. These features correspond with the transition probabilities reported in Table 2.

The parameter values of the economy without aggregate uncertainty are identical to those of the economy with aggregate uncertainty with the following exceptions. Δ^a is set equal to zero and the aggregate capital stock is held constant at 43. The unemployment rate is always equal to μ^b and the transition probabilities are given in Table 3.

6 Exercises for model without aggregate uncertainty

For the model without aggregate uncertainty, the users are asked to report the following.

- Calculate, as a function of the agent's beginning-of-period capital stock, the capital choice for an employed and an unemployed agent and for both aggregate states. Generate capital decisions as a function of capital on a grid where capital varies from 0 to 100 and the stepsize is 0.01 (when $k < 5$) and 0.1 when $k > 5$. Generate a file with five columns. The first column contains the values of k, the second $k'(\varepsilon = 0, a = a^b)$, the third $k'(\varepsilon = 1, a = a^b)$, the fourth $k'(\varepsilon = 0, a = a^g)$, and the fifth $k'(\varepsilon = 1, a = a^g)$.

- Calculate the level of capital at which the zero-capital constraint is just binding for an unemployed agent in both aggregate states.

- Using the realizations for the employment status given on the website and an initial capital stock equal to 43 generate a time path for capital. Report results with in the first column the idiosyncratic shock, in the second the beginning of period capital stock, and in the third column the consumption choice. The first row thus consists of 2, 43, and the consumption choice.

- Calculate Euler equation errors on the following grid. Capital varies from 0 to 100 and the stepsize is 0.01. First calculate consumption using your numerical approximation, $c(s_i)$, where s_i is the vector of state variables at grid point i. Next calculate the conditional expectation and the implied consumption choice, $\bar{c}(s_i)$. The Euler equation error is then equal to

$$\frac{|c(s_i) - \bar{c}(s_i)|}{\bar{c}(s_i)}.$$

  Do this exercise for unemployed and unemployed agents and for both aggregate states. For these four combinations report the maximum and average error across the 10,000 gridpoints. Also indicate at which capital level the maximum is attained. Report errors using the same ordering as indicated in the first bullet point.

IMPORTANT!!! It is important that choices made in the numerical solution for the model without aggregate uncertainty are equal to those made in solving the model with aggregate uncertainty unless it is impossible to do so. This part of the comparison becomes meaningless if a much more accurate implementation for this easy part is used.

7 How to simulate the economy with aggregate uncertainty?

7.1 Stochastic versus non-stochastic

Algorithms differ in terms of what state variables are used to characterize the cross-sectional distribution. Compared with the case of no aggregate uncertainty, this makes it much more difficult if not impossible to compare policy functions. A sensible alternative would be to compare the *properties* of the policy functions along a simulated path. This raises the issue on how to simulate, which is the topic in this section.

One possibility is to use a large finite number of agents and simulate a panel. This raises the question what random number generator to use. It would have to be able to generate a truly large data set (say $10,000$ time periods and $100,000$ agents), be portable across computers, and available in different programming languages. An alternative would be to use a numerical procedure to describe the cross-sectional distribution of the continuum of agents. This would avoid the need for a random number generator and it also would be closer to the model since the model has a continuum of agents and the lack of cross-sectional sampling variation is an important property of the definition of the equilibrium.

There are different ways to avoid cross-sectional sampling variation. One way to do the latter is to approximate the density with a histogram on a very fine grid. Another is to parameterize the cross-sectional distribution with a flexible functional form. My hunch is that non-stochastic procedures are quite accurate and it doesn't matter much which procedure is used *but it is important to use the same one to make comparisons of numerical solutions easier.* The simulation procedure to be used is (basically) the one used already by Michael Reiter.

Note that we still have to specify a sequence of realizations for the aggregate shock. This one is made available at
http://www1.fee.uva.nl/toe/content/people/content/denhaan/datasuite.htm.

7.2 A non-stochastic simulation procedure

Information used. The beginning-of-period t distribution of capital holdings is fully characterized by the following:

- the fraction of unemployed agents with a zero capital stock, $p_t^{u,0}$,
- the fraction of employed agents with a zero capital stock,[1] $p_t^{e,0}$,
- the distribution of capital holdings of unemployed agents with positive capital holdings, and
- the distribution of capital holdings of employed agents with positive capital holdings.

Overview. The goal is to calculate the same information at the beginning of the next period. Besides these four pieces of information regarding the cross-sectional distribution one only needs (i) the realizations of the aggregate shock this period and next period and (ii) the individual policy function.

Grid Construct the following grid and define the begining-of-period distribution of capital as follows.

- $\kappa_0 = 0$ and $\kappa_i = 0.1i$, $i = 1, \cdots, 1000$.

[1]Employed agents never choose a zero capital stock but some unemployed agents that chose a zero capital stock last period have become employed this period.

- Let $p_t^{w,0}$ be the fraction of agents with employment status w with a zero capital stock at the beginning of period t.
- For $i > 0$, let $p_t^{w,i}$ be equal to the mass of agents with a capital stock bigger than κ_{i-1} and less than or equal to κ_i. This mass is assumed to be distributed uniformly between gridpoints.
- We have

$$\sum_{i=0}^{1000} p_t^{u,i} = 1, \quad \sum_{i=0}^{1000} p_t^{e,i} = 1.$$

Denote this beginning-of-period distribution function by $P_t^w(k)$.

The initial distribution is also made availabe at http://www1.fee.uva.nl/toe/content/people/content/denhaan/datasuite.htm.

End-of-period distribution The first step is to calculate the end-of-period distribution of capital.

For the unemployed calculate the level of capital holdings at which the agent chooses κ_i. If we denote this capital level by $x_t^{u,i}$ then it is defined by[2]

$$k'(x_t^{u,i}, \cdot) = \kappa_i. \tag{6}$$

Now compute the end-of-period distribution function at the grid points as

$$F_t^{u,i} = \int_0^{x_t^{u,i}} dP_t^u(k) = \sum_{i=0}^{\overline{i_u}} p_t^{u,i} + \frac{x_t^{u,i} - \kappa_{\overline{i_u}}}{\kappa_{1+\overline{i_u}} - \kappa_{\overline{i_u}}} p_t^{u,\overline{i_u}+1}, \tag{7}$$

where $\overline{i_u} = \bar{i}(x_t^{u,i})$ is the largest value of i such that $\kappa_i \leq x_t^{u,i}$. The second equality follows from the assumption that P_t^u is distributed uniformly between gridpoints.

A similar procedure is used to calculate the end-of-period distribution for the employed.

$$F_t^{e,i} = \int_0^{x_t^{e,i}} dP_t^e(k) = \sum_0^{\overline{i_e}} p_t^{e,i} + \frac{x_t^{e,i} - \kappa_{\overline{i_e}}}{\kappa_{1+\overline{i_e}} - \kappa_{\overline{i_e}}} p_t^{e,\overline{i_e}+1},$$

where $\overline{i_e} = \bar{i}(x_t^{e,i})$ is the largest value of i such that $\kappa_i \leq x_t^{e,i}$.

[2] This is a non-linear problem (and has to be calculated at many nodes) but it should be a well behaved problem.

Next period's beginning-of-period distribution Let $g_{w_t w_{t+1} a_t a_{t+1}}$ stand for the mass of agents with employment status w that have employment status w_{t+1}, conditional on the values of a_t and a_{t+1}. For each combination of values of a_t and a_{t+1} we have

$$g_{u_t u_{t+1} a_t a_{t+1}} + g_{e_t u_{t+1} a_t a_{t+1}} + g_{u_t e_{t+1} a_t a_{t+1}} + g_{e_t e_{t+1} a_t a_{t+1}} = 1. \tag{8}$$

We then have

$$P_{t+1}^{w,i} = \frac{g_{u_t w_{t+1}}}{g_{u_t w_{t+1}} + g_{e_t w_{t+1}}} F_t^{u,i} + \frac{g_{e_t w_{t+1}}}{g_{u_t w_{t+1}} + g_{e_t w_{t+1}}} F_t^{e,i} \tag{9}$$

and

$$p_{t+1}^{w,0} = P_{t+1}^{w,0} \tag{10}$$

$$p_{t+1}^{w,i} = P_{t+1}^{w,i} - P_{t+1}^{w,i-1} \tag{11}$$

Note that to implement this procedure and to ensure that differences in the simulated output is only due to differences in the policy functions used we have to use the same interval length, which is set equal to 0.1. Since setting the upperbound can be an important part of the program, participants are free to set their own upperbound.

8 Exercises for model with aggregate uncertainty

8.1 Properties of individual policy function

Statistics in this section are based on a simulation of the whole economy.[3] This simulation procedure uses *only* the individual policy rules and not any aggregate policy rule. Thus, the aggregate policy rule should also not be used to generate inputs for individual policy rules. Those inputs should be calculated using the simulated data.[4] Any differences in the statistics reported are, thus, necessarily due to differences in the individual policy functions. A comparison of differences in the aggregate laws of motion across numerical procedures is done separately.

When using a procedure without cross-sectional sampling variation, the simulation of the economy only requires a sequence of realizations for the aggregate

[3] As mentioned above, the fact that different algorithms use different sets of state variables makes a direct comparison of individual policy functions difficult.

[4] For example, if the individual policy rule depends on the mean capital stock then the simulated cross-sectional mean should be used, not the mean implied by an aggregate law of motion.

productivity shock. But since also correlations between individual outcomes and aggregate outcomes need to be calculated, we also provide a sequence of realizations of idiosyncratic shocks for one agent. This agent's beginning-of-period capital stock is equal to 43.

8.1.1 Risk sharing

The simulation described above generates a time series with for each period a complete description of the cross-sectional distribution. This is conditional on a time series with realizations of the aggregate shock. To calculate risk sharing properties, the participants are also given a sequence of realizations for one individual agent. The following statistics should be calculated

- correlation of individual and aggregate consumption
- correlation of individual consumption and aggregate income
- correlation of individual consumption and aggregate capital stock
- correlation of individual consumption and individual income
- correlation of individual consumption and individual capital stock
- standard deviation of individual consumption
- standard deviation of individual capital
- autocorrelation of individual consumption (up to three lags)
- autocorrelation of individual capital (up to three lags)
- autocorrelation of consumption growth.

8.1.2 beginning-of-period cross-sectional distribution of capital

- fraction of times an agent is at the constraint
- fraction of times an agent is at the constraint in the good aggregate state
- fraction of times an agent is at the constraint in the bad aggregate state
- average values of the 5th and 10th percentile (unconditional and conditional on aggregate state) of the capital distribution for the employed and unemployed agent. Note that this is for the distribution of zero and non-zero capital holdings.

- average value and standard deviation of moments of the distribution, $n = 1, 2, 3, 4, 5$. Higher-order moments should be scaled. that is, if $m(n)$ is the $n - th$ moment you have to report $(m(n))^{1/n} / m(1)$, $n = 2, 3, 4, 5$. Report the statistics conditional on employment status and for the population.

8.1.3 Time series

- aggregate state (just to check we are doing the same thing)
- individual state (again just to check)
- begining-of-period capital of our one individual (43 in first period/row)
- individual consumption choice
- for the beginning-of-period distribution
  - mean capital stock of the unemployed
  - mean capital stock of the employed
  - rental and wage rate as implied by these two means
  - 5th percentile of the unemployed
  - 10th percentile of the unemployed
  - 5th percentile of the employed
  - 10th percentile of the employed

8.2 Aggregate properties

This section lists a set of statistics for aggregate variables. If possible, the statistics in this section have to be calculated in two different ways. First, as implied by the simulation described above that does *not* use the aggregate policy rules. Second, by only using the aggregate policy rule.

8.2.1 Prices

- Average and standard deviation of rental and wage rate
- Autocorrelation of both prices (up to three lags)

8.2.2 Business cycle statistics

- Standard deviation of aggregate income, aggregate consumption, and aggregate investment
- Standard table of auto and cross correlations (with output) at leads and lags (up to three)

8.3 Accuracy tests

There are many dimensions in which one can test for accuracy. The exercises here focus on the accuracy of the aggregate law of motion. See Den Haan (2007) for a motivation of these tests. Additional accuracy tests, such as the accuracy of the individual policy rules for the case with aggregate uncertainty are left to the participants.

Den Haan (2007) argues that a solid accuracy tests requires the comparison of a simulation based on the proposed aggregate law of motion with a simulation that does not use the aggregate law of motion at all, like the simulation procedure described above.

Again we use beginning-of-period moments (after the idiosyncratic and aggregate shocks have been realized). Let $m_t^{w,1}$ be the mean capital stocks of agents of employment status $w \in \{u, e\}$ conditional of having a positive capital stock and calculated as the cross-sectional average in the simulation. This series for the mean is constructed without using any numerical solution for the aggregate law of motion. Let $\bar{m}_t^{w,1}$ be the corresponding moment based on a simulation with the proposed solution for aggregate law of motion only. That is, as inputs in the aggregate law of motion to calculate $\bar{m}_t^{w,1}$, one cannot use any of the m_t^w values but one has to use the $\bar{m}_t^w$ values.

Those who do not calculate the means for the two groups separately may only report the results for the mean of the whole population, m^1.

The accuracy tests then consists of the following:[5]

- The maximum and average percentage error between the two series
- Repeat the exercise for a particular realization of the aggregate productivity shock, namely one that is in one state for 100 periods and then switches to the other state to remain there for again 100 periods. Again give the plot and the maximum and average percentage errors.

[5]Initial conditions are given in the programs used to do the simulation.

- For both the random and the peculiar realization report the generated data. The first column has the aggregate state, the second and the third the mean capital stocks of the unemployed and employed coming out of the panel simulation and the fourth and fifth column the mean capital stocks calculated using the (approximating) aggregate law of motion.

Table 1: Benchmark calibration

Parameters	β	γ	α	δ	$\bar{l}$	μ	Δ^a
Values	0.99	1	0.36	0.025	1/0.9	0.15	0.01

Table 2: Transition probabilities

s,ε / s',ε'	1-$\Delta^a, 0$	1-$\Delta^a, 1$	1+$\Delta^a, 0$	1+$\Delta^a, 1$
1-$\Delta^a, 0$	0.525	0.35	0.03125	0.09375
1-$\Delta^a, 1$	0.038889	0.836111	0.002083	0.122917
1+$\Delta^a, 0$	0.09375	0.03125	0.291667	0.583333
1+$\Delta^a, 1$	0.009115	0.115885	0.024306	0.850694

!!! Note that in the previous document the bottom left element was 0.009155 while it should have been 0.009115.

Table 3: Transition probabilities model without aggregate uncertainty

ε / ε'	0	1
0	0.6	0.4
1	0.044445	0.955555