Computing Stochastic Dynamic Economic Models with a Large Number of State Variables: A Description and Applications of Smolyak's Method*

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Abstract

We describe Smolyak's algorithm to compute recursive solutions of dynamic economies with a sizeable number of state variables. We show how powerful this method may be in applications by computing the nonlinear recursive solution of an international real business cycle model with a substantial number of countries, complete insurance markets and frictions that impede frictionless international capital flows. In this economy the aggregate state vector includes the distribution of world capital across different countries as well as the exogenous country-specific technology shocks. Models with up to 6 countries, and thus up to 12 continuous state variables, can be computed efficiently with the proposed algorithm. The greatest challenge to successfully implementing the algorithm is finding appropriate bounds for the state variables.

1 Introduction

The stochastic neoclassical growth model has arguably been the most important workhorse in modern macroeconomic analysis. Its open economy counterpart, the international real business cycle model, has been fruitfully applied to study the co-movement of output, investment, and consumption across countries and

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international capital flows between countries. See Backus et al. (1992) as an important example of this literature and Backus et al. (1995) for an overview.

Apart from highly stylistic examples, international real business cycles cannot be solved analytically, and one has to resort to numerical techniques to compute the equilibrium of these models. Even the simplest version of the model with two countries and country-specific, serially correlated technology shocks requires at least three state variables in its recursive formulation. This explains why most of the literature resorts to (log-)linearization of the equilibrium or optimality conditions of a world social planner problem to solve for equilibrium allocations. Since the true equilibrium is unknown, it is hard to assess how accurate numerical approximations of the equilibrium are that rely on these (log-)linear approximations.

In this paper we propose a projection method based on Smolyak's algorithm to compute *globally accurate* solutions to models characterized by a sizeable number of continuous state variables, such as international real business cycle models with a substantial number of countries. Our objectives are threefold: first, we provide an easily accessible general description of our algorithm; second, we show how powerful this method is by numerically solving an international real business cycle model with many countries and international capital market frictions; and third, we assess the quality of our approximations.

We find that our projection method performs quite well for a wide variety of model specifications including models with up to 6 countries (i.e., 12 continuous state variables), specifications that introduce a great deal of curvature into utility and production functions, and models with asymmetries between countries. Our method is also substantially more accurate than a linear approximation of the solution, especially when the exogenous shocks to the economy are large. The greatest challenge to using this method is finding bounds for the state variables which are not violated by the solution of the model. For international real business cycle models, this is most difficult for asymmetric specifications with a large number of countries.

The rest of the paper is organized as follows. Section 2 provides a general description of our projection method, while section 3 describes the international real business cycle model that we solve in this paper. Section 4 presents results, accuracy tests, and running times. Section 5 concludes.

2 Description of the General Method

[Under Revision.]

3 An International Real Business Cycle Model

In this section, we demonstrate how powerful the method developed in the previous section is in solving models with a significant number of continuous state variables. We will solve for the nonlinear solution of an international real business cycle model with N countries. We will confine ourselves to economies in which equilibria are Pareto optimal, so they can be solved by solving an appropriately defined social planner's problem. In addition, we allow for frictions to international capital flows due to adjustment costs but require that these costs leave the optimal decision rules smooth.

3.1 Description of the Model

We consider a world composed of N countries that are subject to technology shocks which contain a country-specific and a common component. Changes in a country's capital stock are subject to quadratic adjustment costs, which inhibit frictionless flows of capital across the N countries.[1] As a consequence the entire distribution of capital stocks, and not only aggregate output, becomes a state variable in the recursive formulation of the social planner's problem associated with this economy. There are complete asset markets, so that the welfare theorems apply. Thus one can solve for competitive equilibrium allocations by solving a social planner's problem for appropriate welfare weights of individual countries.

For a given set of Pareto weights $(\tau^1, ..., \tau^N)$, the social planner solves the problem

$$
\begin{aligned}
& & \max_{\{\{c_t^i, l_t^i, k_{t+1}^i\}_{i=1}^N\}_{t=0}^\infty} E_0 \sum_{i=1}^N \tau^i \sum_t \beta^t u^i(c_t^i, l_t^i) \\
\ln(a_t^i) &=& \rho \ln(a_{t-1}^i) + \sigma(\varepsilon_t^i + \varepsilon_t) \\
k_{t+1}^i &=& (1-\delta)k_t^i + i_t^i \\
\sum_{i=1}^N c_t^i + \sum_{i=1}^N i_t^i - \sum_{i=1}^N \delta k_t^i &=& \sum_{i=1}^N \left(a_t^i f^i(k_t^i, l_t^i) - \frac{\phi}{2} \frac{(k_{t+1}^i - k_t^i)^2}{k_t^i} \right)
\end{aligned}
$$

where $\varepsilon_t, \varepsilon_t^i$ are *iid* standard normal random variables (note: as long as we know the joint distribution of $(a_t^1, \dots a_t^N)$ given $(a_{t-1}^1, \dots a_{t-1}^N)$, we can solve this model) and $\phi \geq 0$ is a scale parameter. In particular, the parameterization nests the case of no adjustment costs, $\phi = 0$. Denote the initial distribution of capital across countries by $(k_0^1, \dots, k_0^N)$, which determines what point on the Pareto frontier (i.e. what vector of Pareto weights) corresponds to a competitive equilibrium. Our algorithm will solve for optimal policies for arbitrary Pareto weights, and thus (indirectly, and if we do the additional step of mapping Pareto weights into initial wealth distribution, directly) for the entire equilibrium manifold.

In general, the state variables for the recursive formulation of the social planner's problem consist of the vector of current exogenous shocks $a = (a^1, \dots, a^N)$ and the vector of endogenous current capital stocks $k = (k^1, \dots, k^N)$. Denote

[1] The quadratic form of the adjustment cost is not crucial for our algorithm to work. Any strictly convex and continuously differentiable adjustment cost function will do.

by $s = (k, a)$ the current state, which is of dimension $2N$. Below we discuss one example where the number of state variables can be reduced to $N + 1$.

The planner's problem in recursive formulation can be written as

$$\begin{aligned} V(k,a) &= \max_{\{c^i, l^i, k^{i\prime}\}} \sum_{i=1}^{N} \tau^i u^i(c^i, l^i) + \beta \int V(k', a') f_a(a') da' \\ \ln(a^{i\prime}) &= \rho \ln(a^i) + \sigma(\varepsilon_i + \varepsilon) \\ \sum_{i=1}^{N} c^i + \sum_{i=1}^{N} k^{i\prime} + \sum_{i=1}^{N} \frac{\phi(k^{i\prime} - k^i)^2}{2k^i} &= \sum_{i=1}^{N} a^i f^i\left(k^i, l^i\right) + \sum_{i=1}^{N} k^i \end{aligned}$$

where again $f_a(a')$ is the density function over a', given a. We will now derive the system of functional equations used to compute this model. We seek functions $C^i(s)$, $L^i(s)$, and $K^{i\prime}(s)$ for $i = 1, \ldots, N$, mapping the current state $s = (k, a)$ into consumption and labor supply of each country today and its capital stock tomorrow. For future reference we define as

$$\begin{aligned} C(s) &= \sum_{i=1}^{N} C^i(s) \\ Y(s) &= \sum_{i=1}^{N} a^i f^i(k^i, L^i(s)) \\ K &= \sum_{i=1}^{N} k^i \\ K'(s) &= (K^{1\prime}(s), \ldots, K^{N\prime}(s)). \end{aligned}$$

Attaching Lagrange multiplier μ to the resource constraint, we find as first order conditions

$$\begin{aligned} \tau^i u_c^i(c^i, l^i) &= \mu \qquad \forall i \\ \frac{\tau^i u_l^i(c^i, l^i)}{-a^i f_l^i(k^i, l^i)} &= \mu \qquad \forall i \\ \frac{\beta \int V_{k^i}(k', a') f_a(a') da'}{1 + \frac{\phi(k^{i\prime} - k^i)}{k^i}} &= \mu \qquad \forall i \end{aligned}$$

where lower case letters attached to functions denote partial derivatives of the function with respect to the corresponding argument. The envelope condition reads as

$$\begin{aligned} V_{k^i}(k,a) &= \mu \left[\left(1 + a^i f_k^i(k^i, l^i\right) + \frac{\phi}{2} \frac{(k^{i\prime} - k^i)(k^{i\prime} + k^i)}{{k^i}^2} \right] \\ &= \tau^i u_c^i(c^i, l^i) \left[\left(1 + a^i f_k^i(k^i, l^i\right) + \frac{\phi}{2} \frac{(k^{i\prime} - k^i)(k^{i\prime} + k^i)}{{k^i}^2} \right] \qquad \forall i \\ &= \tau^j u_c^j(c^j, l^j) \left[\left(1 + a^i f_k^i(k^i, l^i\right) + \frac{\phi}{2} \frac{(k^{i\prime} - k^i)(k^{i\prime} + k^i)}{{k^i}^2} \right] \qquad \forall i, j. \end{aligned}$$

Combining the first order conditions and the envelope conditions gives (replacing choices by policy functions and abusing notation by writing $s' = (K'(s), a')$)

$$\tau^i u_c^i \left(C^i(s), L^i(s)\right) = \frac{\beta \int \tau^i u_c^i \left(C^i\left(s'\right), L^i(s')\right) \left[\begin{array}{c} 1 + a^{i\prime} f_k^i \left(K^{i\prime}, L^i(s')\right) + \\ \frac{\phi}{2} \frac{(K^{i\prime}(s') - K^{i\prime}(s))(K^{i\prime}(s') + K^{i\prime}(s))}{K^{i\prime}(s)^2} \end{array} \right] f_a(a') da'}{1 + \frac{\phi(K^{i\prime}(s') - K^{i\prime}(s))}{K^{i\prime}(s)}} \tag{1}$$

which together with

$$\begin{aligned} \tau^i u_c^i \left(C^i(s), L^i(s)\right) &= \tau^j u_c^j \left(C^j(s), L^j(s)\right) && (2) \\ \frac{u_l^i \left(C^i(s), L^i(s)\right)}{u_c^i \left(C^i(s), L^i(s)\right)} &= -a^i f_l^i(k^i, L^i(s)) && (3) \\ C(s) + \sum_{i=1}^{N} K^{i\prime}(s) + \sum_{i=1}^{N} \frac{\phi(K^{i\prime}(s) - k^i)^2}{2k^i} &= Y(s) + K && (4) \end{aligned}$$

provides $3N$ functional equations to be jointly solved for the $3N$ functions $\{C^i(s), L^i(s), K^{i\prime}(s)\}_{i=1}^N$. We seek approximations to the functions $\{C^i(s), L^i(s), K^{i\prime}(s)\}_{i=1}^N$ of the form given by (**??**).

3.2 Special Cases

In general, these $3N$ functional equations have to be solved jointly, but there are special cases where the problem becomes easier. If labor is supplied inelastically, we can drop the N functional equations (3), leaving $2N$ functional equations to be jointly solved for the $2N$ functions $\{C^i(s), K^{i\prime}(s)\}_{i=1}^N$. There is a sense in which production and consumption decisions are separable. The $N-1$ equations

$$\begin{aligned} &\frac{\beta \int \tau^i u_c^i \left(C^i\left(s'\right)\right) \left[\begin{array}{c} 1 + a^{i\prime} f_k^i \left(K^{i\prime}(s)\right) + \\ \frac{\phi}{2} \frac{(K^{i\prime}(s') - K^{i\prime}(s))(K^{i\prime}(s') + K^{i\prime}(s))}{K^{i\prime}(s)^2} \end{array} \right] f_a(a') da'}{1 + \frac{\phi(K^{i\prime}(s') - K^{i\prime}(s))}{K^{i\prime}(s)}} \\ = &\frac{\beta \int \tau^i u_c^i \left(C^i\left(s'\right)\right) \left[\begin{array}{c} 1 + a^{j\prime} f_k^j \left(K^{i\prime}(s)\right) + \\ \frac{\phi}{2} \frac{(K^{j\prime}(s') - K^{j\prime}(s))(K^{j\prime}(s') + K^{j\prime}(s))}{K^{j\prime}(s)^2} \end{array} \right] f_a(a') da'}{1 + \frac{\phi(K^{j\prime}(s') - K^{j\prime}(s))}{K^{j\prime}(s)}} \end{aligned} \tag{5}$$

for a given total amount of capital to be carried forward to tomorrow determine the allocation of capital across the countries. However, the decision of how much total consumption and how much accumulation is optimal cannot in general be solved for independent of the allocation of consumption across countries, which is simply another way of saying that the $2N$ equations have to solved jointly. Note that so far no assumptions about the functional form of u^i, f^i and the equality of preference or technology parameters have been made.

3.2.1 Exogenous Labor and CRRA Utility

There are two interesting examples for which the problem with exogenous labor supply becomes even easier. Suppose all households have identical CRRA period utility function with risk aversion parameter γ. Then from

$$\tau^i u_c^i \left(C^i(s)\right) = \tau^j u_c^j \left(C^j(s)\right),$$

it follows that

$$C^i(s) = \left(\frac{\tau^i}{\tau^1}\right)^{\frac{1}{\gamma}} C^1(s)$$

and thus

$$C(s) = \sum_{i=1}^{N} C^i(s) = \frac{C^1(s)}{\left(\tau^1\right)^{\frac{1}{\gamma}}} \sum_{i=1}^{N} \left(\tau^i\right)^{\frac{1}{\gamma}}$$

and thus consumption follows the linear risk sharing rule

$$C^i(s) = \frac{\left(\tau^i\right)^{\frac{1}{\gamma}}}{\sum_{i=1}^{N} \left(\tau^i\right)^{\frac{1}{\gamma}}} C(s). \tag{6}$$

That is, each agents' consumption is a constant fraction of aggregate consumption. For this example one can first jointly solve for aggregate consumption, aggregate investment and its allocation across countries, and then separately solve for the distribution of consumption across countries, according to (6). Using (6) in (5) yields the $N-1$ equations

$$\begin{aligned} & \frac{\int \left(C\left(s'\right)\right)^{-\gamma} \left[\begin{array}{c} 1 + a^{i\prime} f_k^i \left(K^{i\prime}(s)\right) + \\ \frac{\phi}{2} \frac{\left(K^{i\prime}(s') - K^{i\prime}(s)\right)\left(K^{i\prime}(s') + K^{i\prime}(s)\right)}{K^{i\prime}(s)^2} \end{array}\right] f_a(a') da'}{1 + \frac{\phi\left(K^{i\prime}(s') - K^{i\prime}(s)\right)}{K^{i\prime}(s)}} \\ = & \frac{\int \left(C\left(s'\right)\right)^{-\gamma} \left[\begin{array}{c} 1 + a^{j\prime} f_k^j \left(K^{i\prime}(s)\right) + \\ \frac{\phi}{2} \frac{\left(K^{j\prime}(s') - K^{j\prime}(s)\right)\left(K^{j\prime}(s') + K^{j\prime}(s)\right)}{K^{j\prime}(s)^2} \end{array}\right] f_a(a') da'}{1 + \frac{\phi\left(K^{j\prime}(s') - K^{j\prime}(s)\right)}{K^{j\prime}(s)}} \end{aligned} \tag{7}$$

which, together with the resource constraint (4) and an equation similar to (1) can be solved for the functions $(C(s), K^{1\prime}(s), \ldots, K^{N\prime}(s))$. Now indeed a complete separation between production and consumption allocations arises.

3.2.2 Exogenous Labor and No Capital Adjustment Cost

If there are no adjustment costs, that is, if $\phi = 0$, then we can reduce the number state variables from the $2N$ variables (k, a) to the $N+1$ variables $s = (y, a)$

where y is total output in the current period.

$$\begin{aligned}
V(s) &= \max_{\{c^i, k^{i\prime}\}} \sum_{i=1}^{N} \tau^i u^i(c^i) + \beta \int V(s') f_a(a') da' \\
\sum_{i=1}^{N} c^i + \sum_{i=1}^{N} k^{i\prime} &= \sum_{i=1}^{N} a^i f^i\left(k^i\right) + \sum_{i=1}^{N} k^i \\
y' &= \sum_{i=1}^{N} a^{i\prime} f^i(k^{i\prime}).
\end{aligned}$$

The first order and envelope conditions imply

$$\mu = \tau^i u_c^i(c^i) = \beta \int \left(1 + a^{i\prime} f_k^i(k^{i\prime})\right) V_y(y'(a'), a') f_a(a') da' = V_y(s).$$

The $2N$ functional equations determining the functions $\{C^i(s), K^{i\prime}(s))\}_{i\in I}$ are the same as in the general case (with $\phi = 0$), but the policy functions are simply functions of the $N+1$ state variables $s = (y, a)$.

3.3 Implementation of the Algorithm

We will approximate policy functions by functions of the form described in (**??**) and use policy function iteration in order to solve for the coefficients determining the policy functions. First we guess policy functions $K_0^{i\prime}(s), L_0^i(s)$ and $C_0^i(s)$ for all $i \in I$. For a given iteration $n-1$ and associated policy functions $\{K_{n-1}^{i\prime}(s), L_{n-1}^i(s), C_{n-1}^i(s)\}_{i\in I}$ the iteration n policy functions are, for all $s = (k, a)$, defined by the $3N$ equations[2]

$$\begin{aligned}
&\tau^i u_c^i\left(C_n^i(s), L_n^i(s)\right) \\
=& \frac{\beta \int \tau^i u_c^i\left(C_{n-1}^i\left(s'\right), L_{n-1}^i(s')\right) \left[\begin{array}{c} 1 + a^{i\prime} f_k^i\left(K_n^{i\prime}(s), L_{n-1}^i(s')\right) + \\ \frac{\phi}{2} \frac{(K_{n-1}^{i\prime}(s') - K_n^{i\prime}(s))(K_{n-1}^{i\prime}(s') + K_n^{i\prime}(s))}{K_n^{i\prime}(s)^2} \end{array}\right] f_a(a') da'}{1 + \frac{\phi(K_{n-1}^{i\prime}(s') - K_n^{i\prime}(s))}{K_n^{i\prime}(s)}} \\
=& \frac{\beta \int \tau^i u_c^i\left(C_{n-1}^i\left(s'\right), L_{n-1}^i(s')\right) \left[\begin{array}{c} 1 + a^{j\prime} f_k^j\left(K_n^{i\prime}(s), L_{n-1}^j(s')\right) + \\ \frac{\phi}{2} \frac{(K_{n-1}^{j\prime}(s') - K_n^{j\prime}(s))(K_{n-1}^{j\prime}(s') + K_n^{j\prime}(s))}{K_n^{j\prime}(s)^2} \end{array}\right] f_a(a') da'}{1 + \frac{\phi(K_{n-1}^{j\prime}(s') - K_n^{j\prime}(s))}{K_n^{j\prime}(s)}} \\
=& \tau^j u_c^j\left(C_n^j(s), L_n^j(s)\right)
\end{aligned}$$

[2] Again we abuse notation and let $s' = (K_n'(s), a')$.

$$\begin{aligned}
\frac{u_l^i\left(C_n^i(s), L_n^i(s)\right)}{u_c^i\left(C_n^i(s), L_n^i(s)\right)} &= -a^i f_l^i(k^i, L_n^i) \\
C_n(s) + \sum_{i=1}^{N} K_n^{i\prime}(s) + \sum_{i=1}^{N} \frac{\phi(K_n^{i\prime}(s) - k^i)^2}{2k^i} &= Y(s) + K.
\end{aligned}$$

Note that it in general cannot be established that the operator defined by these equations is a contraction mapping. In contrast, in a previous paper, Krueger and Kubler (2003), this problem did not arise since the finite horizon of households living in a stochastic OLG economy made backward induction feasible and no fixed point argument had to be made.

4 Numerical Results

We present results – including policy functions plots, approximation errors, and running times – for a number of specifications of the model described in the previous section. Throughout, we approximate the policy functions with a polynomial of total degree 4 (i.e., $q-d=2$). Higher-order approximations yield smaller approximation errors (e.g. choosing $q-d=3$ reduces errors by roughly one decimal point) at the cost of longer running times (the running times are roughly 20 times larger for $q-d=3$ than for $q-d=2$).

4.1 Model Specifications

The various model specifications we solve differ by the number of countries N, the forms of the utility and production functions, and the parameter values chosen for the technology shock process, capital adjustment costs, and utility and production functions. "Problem A" of the JEDC Numerical Methods Comparison Project (Den Haan, Judd, and Juillard (2004)) provides a thorough description of the different specifications and their calibration. Rather than repeating all the details here, Table 1 only lists the parameters which vary across specifications.

4.1.1 Challenges for Solving the Model

A few key issues arise in the application of Smolyak's method, described generally in Section 2, to our specific economic model. The first issue is choosing bounds for the state variables, a necessary step since Smolyak's method is defined over a closed interval, $[-1,1]^{2N}$.[3] For the exogenous state variables, we simply set $[ln(\underline{a}^n), ln(\bar{a}^n)] = [\frac{-tr\ \sigma}{1-\rho}, \frac{tr\ \sigma}{1-\rho}]$, where tr is some positive scalar. For the endogenous state variable, the bounds must be chosen with great care to

[3] Although the state variable generally does not lie in $[-1,1]^{2N}$, it lies in the box $\mathcal{B} = [\underline{k}^1, \bar{k}^1] \times ... \times [\underline{k}^N, \bar{k}^N] \times [ln(\underline{a}^1), ln(\bar{a}^1)] \times ... \times [ln(\underline{a}^N), ln(\bar{a}^N)]$. It is straightforward to use a change of variables to map a state $x \in \mathcal{B}$ to $[-1,1]^{2N}$.

Table 1: **Model Specifications**

Mod	N	Volatility	ϕ	γ	η	χ	μ
A1	2,6	L, H	0.5,10	1	–	–	–
A2	2,6	L, H	0.5,10	0.25, 1	0.1, 1	–	–
A3	2,6	L, H	0.5,10	0.25, 1	–	–	–
A4	2,6	L, H	0.5,10	0.25	–	0.83	-0.2
A5	2,6	L, H	0.5,10	(0.25,1)	–	–	–
A6	2,6	L, H	0.5,10	(0.25,1)	(0.1,1)	–	–
A7	2,4	L, H	0.5,10	(0.25,1)	–	–	–
A8	2,4	L, H	0.5,10	(0.2,0.4)	–	(0.75,0.9)	(-0.3,0.3)

Notes: The functional forms for A1 - A8 are described in Den Haan, Judd, and Juillard (2004). In A1-A4, the countries are symmetric, while in A5-A8, the parameters of the utility and production functions differ across countries. An entry (x, y) for a given parameter ζ indicates that country $n = 1, ..., N$ has parameter $\zeta_n = x + \frac{n-1}{N-1}(y - x)$. Low (L) volatility corresponds to $\rho = 0.8, \sigma = 0.001$ and high (H) volatility to $\rho = 0.95, \sigma = 0.01$.

ensure the capital policy function, $K^{n\prime}(k, a)$, stays in-bounds. All else equal, the wider the capital bounds the more likely the capital policy function is to stay in-bounds but the poorer the accuracy of the solution.

There is also an interaction between the capital and technological shock bounds. As is known from a one-country stochastic growth model, capital exhibits a positive, hump-shaped response to a technological shock, and thus, for a fixed capital interval, larger technological shocks make it more likely for the capital policy function to go out-of-bounds. For this reason, the high volatility case tends to be more difficult to solve than the low volatility case, but both can be solved with the proper choice of bounds. Unless noted otherwise, we use $tr = 1.25$ and $[\underline{k}, \bar{k}] = [0.5k_{ss}, 1.5k_{ss}]$ to generate all the results for the symmetric cases (A1-A4) reported below.

For asymmetric cases with endogenous labor supply (A6-A8), finding appropriate bounds for the capital interval provides an additional challenge because the true capital policy functions are asymmetric across countries. To see this, consider specifications A6-A8 without any capital adjustment costs ($\phi = 0$) or uncertainty ($ln(a^n_{t+1}) = 0$). In equilibrium, the marginal utility of consumption is equalized across countries at each date, and thus, one can see from the intertemporal Euler equations that the distribution of $t + 1$ capital will be set to equate the marginal product of capital across countries in $t + 1$, which requires equating countries' capital/labor ratios. Because some countries have more elastic labor supplies than others, the countries will supply different levels of labor if provided with the same non-steady-state capital stock – Figure 3 (to be described later) illustrates this for specification A6 – and consequently, the true equilibrium features asymmetric capital policy functions. Solving for these policy functions thus requires specifying asymmetric bounds for the capital intervals.[4] In practice, we let $[\underline{k}_1, \bar{k}_1] = [0.45k_{ss}, 1.55k_{ss}]$, $[\underline{k}_N, \bar{k}_N] = [0.55k_{ss}, 1.45k_{ss}]$, and choose intermediate values for the widths of the other countries' capital intervals.

A second issue that arises in using Smolyak's method is finding a good initial guess for the policy functions. As discussed in Section 3.3, it is not possible to establish that the operator used for the time-iteration procedure is a contraction mapping, and thus, convergence is not guaranteed for any initial guess. In fact, it turns out that poor guesses for labor supply often lead to difficulties for our method. To minimize these difficulties, we have at times found it useful to slightly alter the policy function iteration algorithm described in Section 3.3. Rather than using the previous iteration of the policy function for labor supply $L_{n-1}(s')$ on the right-hand side of the intertemporal Euler equation, we solve for tomorrow's labor, denoted by L^*, using

$$\frac{u_l^i(C_{n-1}^i(s'), L^*)}{u_c^i(C_{n-1}^i(s'), L^*)} = -a^i f_l^i(K_n^{i\prime}(s), L^*). \tag{8}$$

This reduces the importance of the initial guess for labor supply, which turns out to be especially helpful for finding a solution in cases A6-A8.

Finally, our procedure can solve specifications of the model with up to $N = 6$ countries (12 state variables). As currently written, our procedure does not exploit any of the symmetries between the countries (for cases A1-A4) when computing the solution of the model. But because the Smolyak points are symmetric (see Figure **??**), one could envision utilizing the symmetry of cases A1-A4 by solving for the policy functions of only one country at each iteration and then doing the proper transformation to generate the policy functions of the other countries. Doing so may make solving for $N = 10$ countries feasible in these cases. However, given the results reported below, in which increasing the number of countries does not appear to lead to particularly interesting economic insights, we choose not to pursue this direction of research at this point.

4.1.2 Policy Functions

Figures 1 - 3 plot the country-specific capital, consumption, and labor policy functions for specification A6 with $N = 2$, high volatility, and low adjustment costs. There are four plots in each figure: the plots show the policy functions of both countries (blue-solid = country 1, red-dashed = country 2) as a function of own and other capital stock (top two plots) and own and other technology shock (bottom two plots), holding the other state variables at their steady-state values.[5]

[4] The program can converge when using symmetric capital intervals (for example, symmetric intervals were used to create Figures 1 - 3), but the accuracy is better in the case of asymmetric capital intervals.

[5] In order to show all the interesting movements in the policy functions, we solved over a capital interval of $[0.1k_{ss}, 1.9k_{ss}]$ and set $tr = 1.25$ so the technological shock interval is [-0.25, 0.25].

These policy functions are representative, in a qualitative sense, of the policy functions from other specifications. In particular, the capital policy functions are fairly linear in all state variables, while consumption is slightly concave with respect to capital and slightly convex with respect to technology shocks. The labor supply functions clearly display the most significant non-linearities. We take these non-linearities as evidence that non-linear solution methods do, in fact, provide better approximations of the true solution of the model than linear methods. Kim, Kim, and Kollmann (2007) (KKK) document this same result quantitatively by showing that a quadratic approximation of the solution outperforms a linear approximation. Our solution method is also significantly more accurate than a linear approximation, as can be seen by comparing our results reported below (in Table 2) to the linear approximation results in KKK.

It is also interesting to note the effect of asymmetric parameter values on the policy functions. The labor supply functions again display the most interesting results. The equilibrium condition for labor supply of country n can be written as

$$l_t^n = \left[\frac{A^{1/\gamma^1} a_t^n {k_t^n}^{\alpha}}{{c_t^1}^{1/\gamma^1}} \right]^{\frac{\eta^n}{1+\eta^n \alpha}},$$

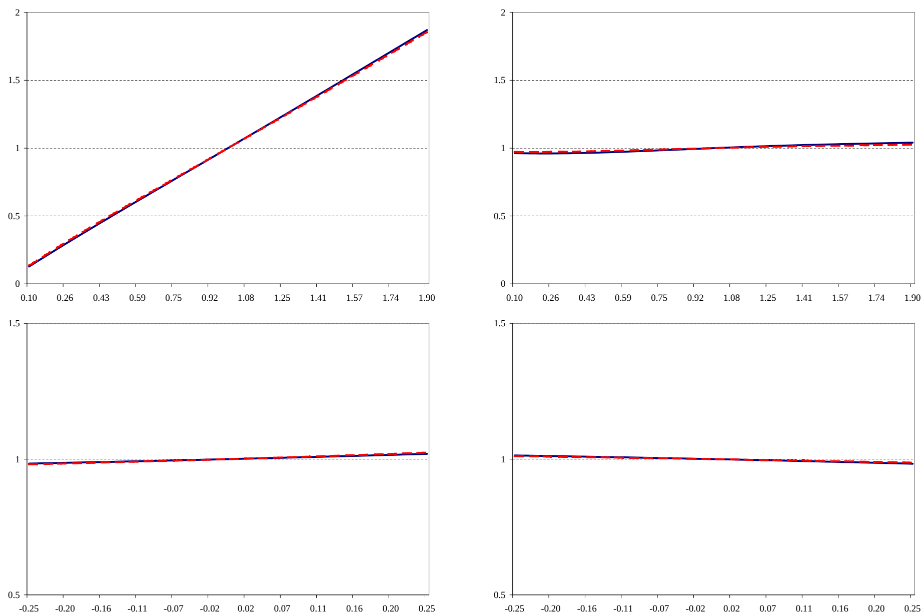
where we have substituted in the equilibrium condition equating the marginal utility of consumption across countries.[6] Thus, it is easy to see that country 2 ($\eta^2 = 1.0$) will have a more elastic labor supply (with respect to a movement in productivity) than country 1 ($\eta^1 = 0.1$), as is confirmed by Figure 3. It is also interesting to note that, except at very low capital levels, labor supply falls with an increase in own-country capital because the income effect (i.e., increased consumption) overwhelms the effect of a higher wage (top-left panel of Figure 3). Furthermore, consumption is also more elastic for country 2 ($\gamma^2 = 1.0$) than country 1 ($\gamma^1 = 0.25$) because the equilibrium condition for consumption-sharing implies that c_t^2 is proportional to ${c_t^1}^{\gamma^2/\gamma^1}$.

4.2 Approximation Errors and Running Times

We check the accuracy of the solution in three ways, the first two of which require the computation of conditional error functions. These functions, denoted by $R_i(x_t) = 0$, for $i = 1, .., 3N$, are unit-free versions of the $3N$ equilibrium

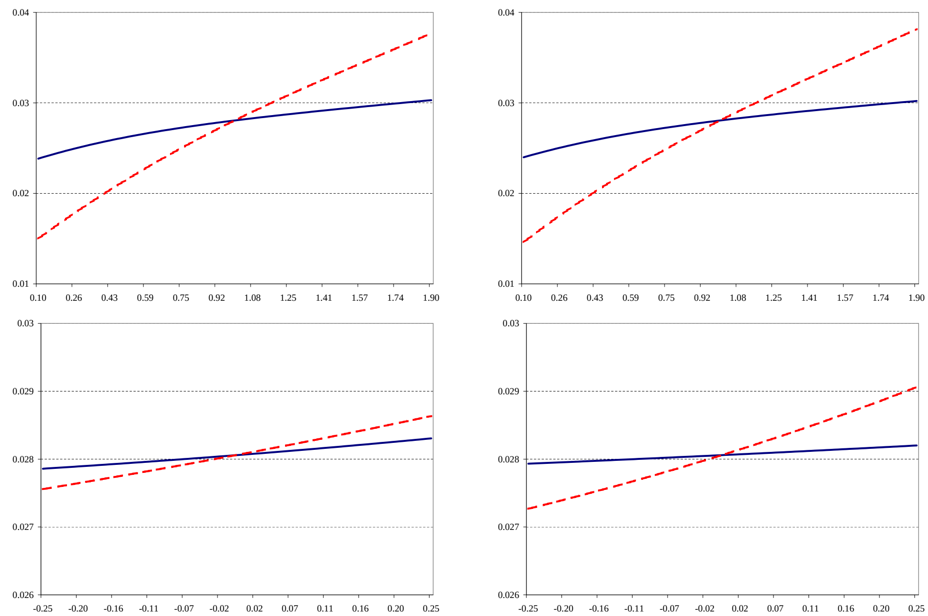
[6]Note that the Pareto weights are $\tau_n = \frac{1}{u_c^n(c_{ss}^n, l_{ss}^n)} = A^{1/\gamma^n}$.

Figure 1: Capital Policy Functions for A6



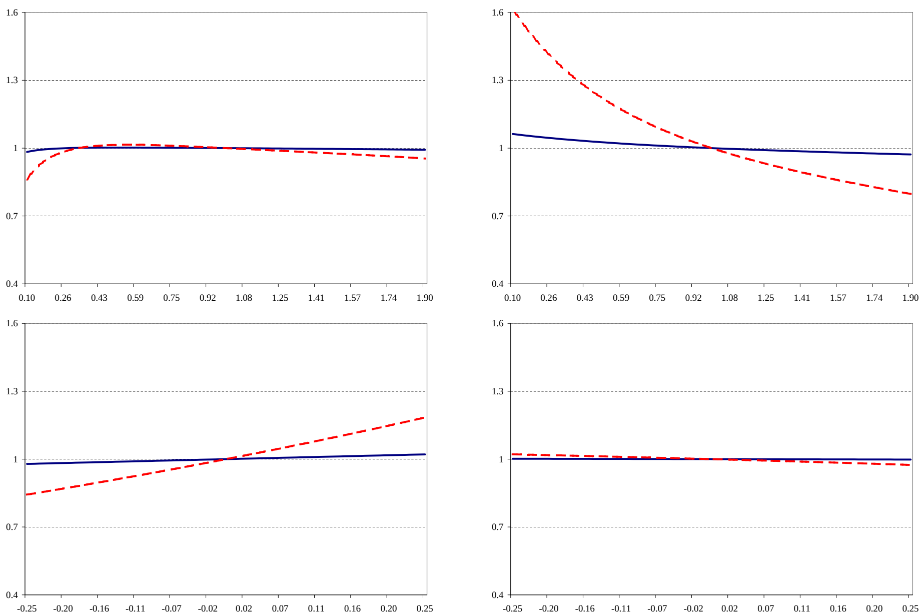
Notes: Capital stock tomorrow as a function of own and other capital stocks (top two plots) and own and other technology shocks (bottom two plots), holding other state variables at steady-state values. The blue lines are for country 1 ($\gamma = 0.25, \eta = 0.1$), and the red-dashed lines are for country 2 ($\gamma = 1.0, \eta = 1.0$). Model specification: A6, $N = 2$, high volatility, $\phi = 0.5$.

Figure 2: Consumption Policy Functions for A6



Notes: Consumption as a function of own and other capital stocks (top two plots) and own and other technology shocks (bottom two plots), holding other state variables at steady-state values. The blue lines are for country 1 ($\gamma = 0.25, \eta = 0.1$), and the red-dashed lines are for country 2 ($\gamma = 1.0, \eta = 1.0$). Model specification: A6, $N = 2$, high volatility, $\phi = 0.5$.

Figure 3: Labor Policy Functions for A6



Notes: Labor supply as a function of own and other capital stocks (top two plots) and own and other technology shocks (bottom two plots), holding other state variables at steady-state values. The blue lines are for country 1 ($\gamma = 0.25, \eta = 0.1$), and the red-dashed lines are for country 2 ($\gamma = 1.0, \eta = 1.0$). Model specification: A6, $N = 2$, high volatility, $\phi = 0.5$.

conditions evaluated at the present state $x_t \equiv (k_t, a_t)$:

$$
\begin{aligned}
\beta E_t \left\{ \frac{u^n_{c,t+1}}{u^n_{c,t}} \frac{\left[1 + a^n_{t+1} f^n_{k,t+1} + \phi^n_{2,t+1}\right]}{\left[1 + \phi^n_{1,t}\right]} \right\} - 1 &= 0, \text{ for } n = 1, ..., N \\
\frac{\tau^n u^n_{c,t}}{\tau^1 u^1_{c,t}} - 1 &= 0, \text{ for } n = 2, ..., N \\
\frac{u^n_{c,t} a^n_t f^n_{l,t}}{u^n_{l,t}} + 1 &= 0, \text{ for } n = 1, ..., N \\
\frac{Y_t + K_t}{C_t + K_{t+1} + \phi_t} - 1 &= 0.
\end{aligned}
$$

Letting $R(x_t)$ denote the $3N$-dimensional vector of conditional errors evaluated at state, x_t, Accuracy Tests 1 and 2 are implemented as follows:

Accuracy Test 1: $R(x_t)$ is computed for 100 independent random vectors $x_t = \{k^i_t, a^i_t\}^N_{i=1}$ at radius r from the deterministic steady-state, for $r = \{0.01, 0.02, 0.05, 0.10, 0.15, 0.20, 0.30\}$.[7] We report $T_r \equiv max_{i,t}|R_{i,t}|$.

Accuracy Test 2: The model is simulated for 1000 periods.[8] Let $S_{i,max} \equiv max_t|R_{i,t}|$ and $S_{i,mean} \equiv mean(|R_{i,t}|)$ for $t = 1, ..., 1000$. We report the maximum (across i) $S_{i,max}$ and $S_{i,mean}$, denoted by S_{max} and S_{mean}, respectively.

The third accuracy test is the so-called Den Haan - Marcet statistic (Den Haan and Marcet (1994)) and focuses solely on the N intertemporal equilibrium conditions. This statistic tests whether the *realized* errors in the intertemporal conditions are orthogonal to a constant and first- and second-order monomials of the state variables. This holds for the true solution since the intertemporal equilibrium conditions are conditional expectations. Accuracy Test 3 is implemented as follows:

Accuracy Test 3: We run 200 simulations of the model, each lasting 1000 periods. For each run, the Den Haan - Marcet statistic (1994, p.5) is constructed, and we compare the frequency distribution of this statistic (across the 200 simulations) to the theoretical $\chi^2_{N(2N^2+3N+1)}$ distribution, where the degrees of freedom for the χ^2 distribution are determined by the number of intertemporal equilibrium conditions (N) multiplied by the number of instruments ($2N^2 + 3N + 1$). We report the percentage of the simulations with a

[7]Recall that our solution method requires us to place bounds on the state variables. The bounds for the capital stocks are roughly 0.5 units from the steady state, while those for the technology shocks are 0.25 units away ($tr = 1.25$, $\rho = 0.95$, $\sigma = 0.01$). Thus, it is possible that a sampled point $r = 0.30$ units from the steady state could lie outside the technology shock bound for one country. In this case, we reduce the deviation from the steady state in that dimension and increase the deviations equally in all other dimensions. In effect, we choose a different sample point that is still $r = 0.30$ units from the steady-state and also in-bounds.

[8]The state variables are initially set at their steady state values. The actual length of the simulation was 1200 periods, and the first 200 periods were discarded to ensure independence from initial conditions. We use this same 'burn-off' period for Accuracy Test 3.

statistic in the lower $[P_{.05}]$ and upper $[P_{.95}]$ critical 5% of the χ^2 distribution. Fairly similar distributions are taken as evidence for an accurate solution.

Table 2 reports results of the accuracy tests for 20 individual specifications: for each of A1-A8, we set N at its smallest and largest value given in Table 1; for A2 and A3 (each N), we consider the specification of the utility function with the greatest and least curvature; and in all cases, we choose high volatility, $\rho = 0.95$ and $\sigma = 0.01$, and low adjustment costs $\phi = 0.5$. The reason we report results only for the high volatility and low adjustment cost cases is that these cases usually[9] have the largest conditional error functions. Increasing the adjustment costs to $\phi = 10$ typically reduces the errors only slightly, while lowering the volatility has a much larger effect. In fact, of all parameters, those with the biggest impact on the solution accuracy are the volatility parameters.

From Table 2, it is clear that the accuracy is highest when the economy is close to the steady state, as T_r is increasing in r. This fact also helps explain why the errors reported for Accuracy Test 2 lie between the errors for $T_{.01}$ and $T_{.3}$: in the particular simulation we ran for Accuracy Test 2, the state variables $[k, ln(a)]$ always lie within 0.1 of their steady-state values.

One can also see from Table 2 that the number of countries N does not have a large impact on the error measures T_r, S_{max}, and S_{mean}. The curvature parameters have slightly more of an impact as specifications with more curvature (low η and/or low γ for A2-A3) have larger approximation errors close to the deterministic steady-state ($T_{.01}$) but smaller approximation errors further away ($T_{.30}$). Finally, functional forms A3/A7 and A4/A8 appear to have slightly larger approximation errors than A1/A5 and A2/A6, although this may have as much to do with differences in curvature (parameter values) as it does with the particular functional forms.

It turns out that the largest errors of the conditional error functions almost always correspond to the intertemporal Euler equations. This deserves some comment. In our solution procedure, the static conditions that determine labor supply and the sharing of consumption across countries hold quite exactly. This is because we solve these equations as functions of the state variables and consumption of country 1, without imposing any functional form on the labor supply of any country or the consumption of countries $2-N$. Thus, even though the policy functions for labor supply may be highly nonlinear (see top-left panel of Figure 3), they do not present a problem for our solution method. Rather, any approximation errors will occur mainly in the intertemporal Euler equations and aggregate resource constraint.

The Den Haan - Marcet accuracy measures in Table 2 are much worse when the number of countries is large ($N = 4, 6$) than when $N = 2$: the percentage of observations of the test statistic in the upper critical 5% of the χ^2 distribution

[9] The high volatility case of a particular specification always has larger errors than the low volatility case, while the low adjustment cost case usually does. In the instances when high adjustment costs produce larger errors, the difference is never greater than 0.25 (in log_{10}). None of our qualitative conclusions hinge importantly on reporting results for $\phi = 0.5$ rather than $\phi = 10$.

Table 2: **Accuracy tests: results for selected specifications**

	γ	η	**Test 1** $\mathbf{T_{.01}}$	$\mathbf{T_{.1}}$	$\mathbf{T_{.3}}$	**Test 2** $\mathbf{S_{max}}$	$\mathbf{S_{mean}}$	**Test 3** $\mathbf{P_{.05}}$	$\mathbf{P_{.95}}$
N = 2									
A1	1.0	–	-6.0	-5.1	-4.2	-5.2	-5.8	.04	.06
A2	0.25	0.1	-5.4	-4.7	-4.1	-4.7	-5.3	.03	.09
A2	1.0	1.0	-6.0	-4.8	-3.7	-4.6	-5.5	.03	.08
A3	0.25	–	-5.3	-4.3	-3.7	-4.2	-5.0	.03	.07
A3	1.0	–	-5.9	-4.6	-3.7	-4.4	-5.3	.03	.06
A4	0.25	–	-5.2	-4.3	-3.7	-4.3	-4.9	.03	.07
A5	(.25,1)	–	-5.8	-4.9	-4.0	-5.1	-5.6	.04	.06
A6	(.25,1)	(.1,1)	-5.8	-4.7	-3.9	-4.9	-5.6	.04	.06
A7	(.25,1)	–	-5.3	-4.3	-3.6	-4.2	-4.8	.02	.07
A8	(.2,.4)	–	-4.9	-3.9	-3.1	-3.8	-4.5	.02	.07
N = 6 for A1-A6 , N = 4 for A7-A8									
A1	1.0	–	-5.9	-5.2	-4.6	-5.1	-5.8	0	.47
A2	0.25	0.1	-5.3	-4.8	-4.5	-4.7	-5.3	0	.51
A2	1.0	1.0	-6.2	-4.9	-4.0	-4.4	-5.3	0	.58
A3	0.25	–	-5.4	-4.6	-4.0	-4.1	-4.8	0	.55
A3	1.0	–	-5.9	-4.7	-3.7	-4.0	-5.0	0	.64
A4	0.25	–	-5.3	-4.2	-3.1	-3.4	-4.0	0	.55
A5	(.25,1)	–	-5.5	-5.1	-4.6	-5.0	-5.6	0	.48
A6	(.25,1)	(.1,1)	-4.4	-4.2	-3.8	-4.3	-4.4	0	.53
A7	(.25,1)	–	–	–	–	–	–	–	–
A8	(.2,.4)	–	-4.9	-4.0	-3.6	-4.1	-4.6	0	.38

Notes: The first three columns specify the model and some parameters that vary across alternative specifications. All reported statistics are for high volatility, $\rho = 0.95$ and $\sigma = 0.01$, and low adjustment costs $\phi = 0.5$. The figures shown for Tests 1 and 2 are logs of the error measures $(log_{10}(T_r), log_{10}(S_{max}), log_{10}(S_{mean}))$.

is much greater than 5% for large N. At this point, it is not clear whether these measures reflect poor accuracy of our solution method or sensitivity of the DM statistic to a large number of instruments. In Table 2, there are 15 instruments when $N = 2$, 45 when $N = 4$, and 91 when $N = 6$, whereas the largest number of instruments used by Den Haan - Marcet (1994) was 7. Further experiments will be done to assess whether Accuracy Test 3 is, in fact, a good accuracy measure for this class of problems.

Table 3 reports the time required to compute and run accuracy tests on the solutions of different specifications of the model. In particular, the column labelled "Sol" shows the time it takes to compute the solution. For $N = 2$ countries, this is on the order of seconds; for $N = 4$ (not reported), it takes minutes; and for $N = 6$, the program takes a few hours.

Some specifications, namely A7 and A8, take significantly longer to run than others. For these specifications, our solution procedure, as described earlier, solves a nonlinear equation for the labor supply on the right-hand side of the intertemporal Euler equation rather than simply using the labor supply policy function from the previous iteration. Because this must be done quite often, the program takes much more time to converge, and we choose to only solve up to $N = 4$ for these specifications. We also solve a nonlinear equation for labor in specification A6, but because it is possible to solve the equation analytically in this case, the program runs relatively quickly.

5 Conclusion

[To be Added]

Table 3: **Computing Times for selected models**

	γ	η	**Time(seconds)** Sol.	**Test 1**	**Test 2**	**Test 3**
N = 2						
A1	1.0	–	1.0	0.01	0.07	1.9
A2	0.25	0.1	1.7	0.02	0.08	2.2
A2	1.0	1.0	1.5	0.02	0.08	2.2
A3	0.25	–	8.3	0.7	1.0	14.6
A3	1.0	–	6.2	0.7	1.0	14.0
A4	0.25	–	5.1	0.8	1.2	16.9
A5	(.25,1)	–	1.3	0.01	0.06	1.9
A6	(.25,1)	(.1,1)	2.7	0.02	0.08	2.2
A7	(.25,1)	–	116	0.7	1.0	14.4
A8	(.2,.4)	–	79	0.8	1.2	17.2
N = 6 for A1-A6 , N = 4 for A7-A8						
A1	1.0	–	1398	1.6	2.4	1016
A2	0.25	0.1	2068	1.7	2.6	990
A2	1.0	1.0	1999	1.7	2.6	1187
A3	0.25	–	5430	16.7	24.8	1210
A3	1.0	–	4263	16.4	24.4	1025
A4	0.25	–	3326	19.9	28.9	1227
A5	(.25,1)	–	1718	1.6	2.4	1148
A6	(.25,1)	(.1,1)	2061	1.8	2.7	1174
A7	(.25,1)	–	–	–	–	–
A8	(.2,.4)	–	2917	5.6	8.2	70

Notes: Column labelled **"Sol."** is the computing time for the solution of the model, while the columns labelled **"Test 1"**, **"Test 2"**, and **"Test 3"** record the computing time for the various accuracy tests. All reported statistics are for high volatility, $\rho = 0.95$ and $\sigma = 0.01$, and low adjustment costs $\phi = 0.5$.

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