Low-Order Perturbation Analysis of a Multi-Country Complete Markets Model

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This paper solves the multi-country RBC model defined in "Problem A" of den Haan, Judd and Juillard (2007), using the Sims (2000) algorithm that is based on second-order Taylor expansions of the equilibrium conditions. The algorithm is markedly more accurate than linear approximations when exogenous shocks are big.

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1. Introduction

This paper solves the multi-country RBC model described in "Problem A" of the JEDC Numerical Methods Comparison Project (den Haan, Judd and Juillard (2007) [DJJ]) using Chris Sims' MATLAB program gensys2 which implements the Sims (2000) algorithm that is based on second-order Taylor expansions of the equilibrium conditions. Other solutions techniques for dynamic models based on expansions of second (or higher) order have been developed by Guu and Judd (1993), Gaspar and Judd (1996), Judd (1998), Collard and Juillard (2001), Kim and Kim (2003), Schmitt-Grohé and Uribe (2004), Anderson and Levin (2002), Schaumburg (2002), Sutherland (2002), Kollmann (2003b) and Lombardo and Sutherland (2004).

In contrast to the linear, certainty-equivalent approximations that are widely used in macroeconomics, second-order approximations allow to capture the effect of risk on the mean values of endogenous variables. Compared to other non-linear techniques (see Judd (1998)), second-order approximations have two key advantages: the ease with which they can be applied to models with a large number of state variables, and their high computational speed. This explains why a rapidly growing number of studies apply second-order accurate solution algorithms. Sims' gensys2 program is often used in these studies; see, e.g., Bergin and Tchakarov (2003), Kim and Kim (2001), Kim (2003, 2004), Kollmann (2002, 2003a, 2004a,b; 2007), Marzo (2003, 2004), Shin (2004a,b), Straub and Tchakarov (2005), Teo (2003), and Tchakarov (2002), who all study the welfare effects of alternative macroeconomic policy rules/regimes.

So far, there has been little systematic evaluation of the accuracy of solution methods based on second-order approximations. Together with other studies in the JEDC Numerical Methods Comparison Project (Anderson (2004), Jin and Judd (2004), Juillard (2004), Kim et al. (2004)), this paper fills that gap. Its main finding is that (for DJJ Problem A), the Sims algorithm is markedly more accurate than linear approximations when exogenous shocks are big.

2. Equilibrium

In normalized form (see DJJ), the model is defined by the following equations, for countries $n=1,\dots,N$:

$$(\tau^n u^n_{c,t}-\lambda_t)/(\tau^n u^n_{c,t})=0\,, \tag{1}$$

$$(\lambda_t a^n_t f^n_{l,t}+\tau^n u^n_{l,t})/(\tau^n u^n_{l,t})=0\,, \tag{2}$$

$$E_t\left\{\beta\,(\lambda^n_{t+1}/\lambda^n_t)[a^n_{t+1}f^n_{k,t+1}+1+\varphi[1+\tfrac{1}{2}(i^n_{t+1}/k^n_{t+1}-\delta)](i^n_{t+1}/k^n_{t+1}-\delta)]/[1+\varphi(i^n_t/k^n_t-\delta)]\right\}-1=0\,, \tag{3}$$

$$\left(\sum_{n=1}^{N} c^n_t+i^n_t-\delta k^n_t-a^n_t f^n(k^n_t,l^n_t)+(\varphi/2)k^n_t(i^n_t/k^n_t-\delta)^2\right)\Big/\left(\sum_{n=1}^{N} c^n_t+i^n_t-\delta k^n_t\right)=0\,. \tag{4}$$

$$(k^n_{t+1}-i^n_t-(1-\delta)k^n_t)/k^n_t=0 \tag{5}$$

$$\{a^n_t-\exp[\rho\ln(a^n_{t-1})+\sigma(e_t+e^n_t)]\}/a^n_t=0\,, \tag{6}$$

with $e_{t+1},e^n_{t+1}\sim N(0,1)$, $u^n_{c,t}\equiv\partial u^n(c^n_t,l^n_t)/\partial c^n_t$, $u^n_{l,t}\equiv\partial u^n(c^n_t,l^n_t)/\partial l^n_t$, $f^n_{l,t}\equiv\partial f^n(k^n_t,l^n_t)/\partial l^n_t$, $f^n_{k,t}\equiv\partial f^n(k^n_t,l^n_t)/\partial k^n_t$.

3. Solution method [1]

The Sims (2000) algorithm can be applied to models of the following form:

$$E_t \underset{q\times 1}{G}(\underset{q\times 1}{\omega_{t+1}}, \omega_t, \underset{m\times 1}{\varepsilon_{t+1}}) = 0, \tag{7}$$

where ω_t is a vector of variables known at date t, ε_{t+1} is a vector of date t+1 exogenous disturbances with $E_t\varepsilon_{t+1} = 0$ and $E_t\varepsilon_{t+1}\varepsilon'_{t+1} = \Omega$. As pointed out by Sims (2000), the solution of (7) has the form

$$y_{t+1} = F(y_t, \varepsilon_{t+1}), \; x_{t+1} = M(y_{t+1}), \tag{8}$$

where y_t and x_t are linear functions of ω_t: $(y'_t \; x'_t)' = Z\omega_t$, for some square, non-singular matrix Z. Note that the solution can be expressed as

$$\omega_{t+1} = \Psi(\omega_t, \varepsilon_{t+1}) \equiv Z^{-1}\begin{bmatrix} F(Z_1\omega_t, \varepsilon_{t+1}) \\ M(F(Z_1\omega_t, \varepsilon_{t+1})) \end{bmatrix}, \tag{9}$$

where Z_1 is the matrix (consisting of the first rows of *Z*) such that $y_t = Z_1\omega_t$.

Sims (2000) presents an algorithm (and a MATLAB program, gensys2) that constructs 2nd degree polynomials which approximate (8), in the neighborhood of the (deterministic) steady state given by $G(\omega,\omega,0)=0$. The coefficients of those polynomials are functions of Ω and of the first and second derivatives of $G(\omega_{t+1}, \omega_t, \varepsilon_{t+1})$ (evaluated at the steady state). Let $y_{t+1} = \widehat{F}(y_t, \varepsilon_{t+1})$, $x_{t+1} = \widehat{M}(y_{t+1})$ denote the polynomials that approximate (8), and

$$\omega_{t+1} = \widehat{\Psi}(\omega_t, \varepsilon_{t+1}) \equiv Z^{-1}\begin{bmatrix} \widehat{F}(Z_1\omega_t, \varepsilon_{t+1}) \\ \widehat{M}(\widehat{F}(Z_1\omega_t, \varepsilon_{t+1})) \end{bmatrix}. \tag{10}$$

Application to Problem A

(1)-(6) can be written like (7), using $\omega_t = (\ln(\lambda_t), \ln(c_t^1),..,\ln(c_t^N); \ln(l_t^1),.., \ln(l_t^N); \ln(i_t^1),..,\ln(i_t^n); \ln(k_{t+1}^1),..,\ln(k_{t+1}^N);$ $\ln(a_t^1),..,\ln(a_t^N))$; $\varepsilon_{t+1} \equiv (e_{t+1}; e_{t+1}^1,.., e_{t+1}^N)$; $\Omega \equiv \sigma^2 I_{1+N}$ (I_{1+N}: identity matrix with $1+N$ elements).

We use a two-point finite difference procedure (Fackler and Miranda (2002); pp.98, 102) to compute the first and second derivatives of $G(\omega_{t+1}, \omega_t, \varepsilon_{t+1})$ (at the steady state).

The accuracy checks discussed below require to formulate the solution as a "policy function" that expresses the date $t+1$ decision variables as a function of the capital stocks at the beginning of $t+1$, and of productivity at $t+1$ (in the N countries). Let $K_t \equiv (\ln(k_t^1),..,\ln(k_t^N))$, $A_t \equiv (\ln(a_t^1),..,\ln(a_t^N))$, $S_t \equiv (\ln(\lambda_t), \ln(c_t^1),..,\ln(c_t^N); \ln(L_t^1),..,\ln(L_t^N); \ln(J_t^1),..,\ln(J_t^n))$. As $\omega_t = (S_t, K_{t+1}, \mathrm{A}_t)$, the approximate solution (10) can be written as: $\omega_{t+1} = \widehat{\Psi}((S_t, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1})$. For the model here, we verified that (10) has these properties: (i) the "jump variables" S_t have zero influence on ω_{t+1}; (ii) the influence of A_t and ε_{t+1} on ω_{t+1} can be subsumed by A_{t+1}. Thus, the approximate solution can be written as the following policy function: $\omega_{t+1} = \hat{\Xi}(K_{t+1}, \mathrm{A}_{t+1})$. [2]

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[1] For a more detailed presentation of the Sims algorithm, see Kim et al. (2003).

[2] From (i): $\hat{\Psi}((0, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1}) = \hat{\Psi}((S_t, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1})$. From (6): $A_{t+1} = \rho A_t + \Lambda\varepsilon_{t+1}$, where Λ is a matrix. It appears that $\hat{\Psi}((0, K_{t+1}, \mathrm{A}_t), \varepsilon_{t+1}) = \hat{\Psi}((0, K_{t+1}, \tilde{\mathrm{A}}_t), \tilde{\varepsilon}_{t+1}) \;\forall A_t, \varepsilon_{t+1}, \tilde{A}_t, \tilde{\varepsilon}_{t+1}$ with $\rho A_t + \Lambda\varepsilon_{t+1} = \rho\tilde{A}_t + \Lambda\tilde{\varepsilon}_{t+1}$. Thus $\omega_{t+1} = \hat{\Xi}(K_{t+1}, \mathrm{A}_{t+1}) \equiv$ $\hat{\Psi}((0, K_{t+1}, \mathrm{A}_{t+1}/\rho), 0)$.

4. Accuracy checks

Let Ξ be the exact policy function that is approximated by $\hat{\Xi}$. Ψ and Ξ satisfy the condition $E_t G(\Psi(\Xi(K_t,A_t),\varepsilon_{t+1}),\Xi(K_t,A_t),\varepsilon_{t+1})=0$. The accuracy tests evaluate how closely the approximate solution $\hat{\Psi},\hat{\Xi}$ meets this criterion. Let

$$\hat{R}_t(K_t,A_t) \equiv E_t G(\hat{\Psi}(\hat{\Xi}(K_t,A_t),\varepsilon_{t+1}),\hat{\Xi}(K_t,A_t),\varepsilon_{t+1}) \tag{12}$$

be the "conditional error function" of $\hat{\Psi},\hat{\Xi}$. $\hat{R}_t(K_t,A_t)$ is a vector with $6N$ elements (the model has $6N$ equations); let $\hat{R}_{i,t}(K_t,A_t)$ denote the i-th element of $\hat{R}_t(K_t,A_t)$. We compute the expectation in (12) using the monomial integration formula of degree 3 in Judd (1998, p.275).

Accuracy test 1: $\hat{R}_t(K_t,A_t)$ is computed for 100 independent random vectors (K_t,A_t) at radius r from the steady state, for $r\in\{0.01, 0.10, 0.30\}$. [3] We report $T_r \equiv \max_{i,t}|\hat{R}_{i,t}|$.

Accuracy test 2: The model is simulated over 1000 periods (using (10)).[4] We compute $\hat{S}_t \equiv \max_i|\hat{R}_{i,t}|$ for $t\in\mathbf{T}\equiv\{1,10,20,30,..,1000\}$ and report the maximum and the mean of $\hat{S}_t$ (across $\mathbf{T}$), denoted by $S_{\max}$ and S_{mean}, respectively.

Accuracy test 3: 200 simulation runs of 1000 periods are generated. Let $\hat{g}_{t+1}\equiv G(\hat{\omega}_{t+1},\hat{\omega}_t,\varepsilon_{t+1})$, where $\{\hat{\omega}_t\}$ is the simulated series. For each run, we use a statistic described by den Haan and Marcet (1994, p.5) [DM] to test whether the errors $\hat{g}_{i,t+1}$ in expectational equations (1),(5) are orthogonal to a constant, the elements of K_{t+1} and A_t, and to cross-products of those elements. Under the null hypothesis that the numerical solution is exact, the DM statistic has a χ^2 distribution. We compare the frequency distribution of the DM statistic (across the simulation runs) to the theoretical χ^2 distribution; DM argue that a close match between the two distributions indicates high solution accuracy. $P_{0.05}$ $[P_{0.50}]$ $\{P_{0.95}\}$ denote fractions of the simulated DM statistics below the 5% [50%] {95%} critical values of the χ^2 distribution.

5. Results

DJJ consider 84 different specifications of Problem A. We solved the model for each case. **Table 1** summarizes the results for the full set of specifications, as well as for subsets of specifications. For each (sub)set, we report the maximum of T_r and of $S_{\max}$, and the average of S_{mean} across the individual specifications (included in that set). [5] Cols. 1-5 and Cols. 6-10 of the Table show results for a linear model solution and for the quadratic (second-order accurate) solution, respectively (the linear solution is obtained by just using the first-order terms of (10)). It seems interesting to compare these two solutions, as linear solutions have widely been used in macroeconomics. Below, $T_r^L, S_{\max}^L, S_{mean}^L$, $\{T_r^Q, S_{\max}^Q, S_{mean}^Q\}$ refer to the linear {quadratic} approximation.

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[3]We generate the random K_t,A_t as follows: let h be a column vector with $2N$ i.i.d. elements, and $\tilde{h}\equiv rh/(h'\cdot h)^{.5}$ (NB $(\tilde{h}'\cdot\tilde{h})^{.5}=r$); we set $(K_t,A_t)=\log(1+\tilde{h})$ (steady state logged capital and productivity are zero).

[4] All simulations use the Kim et al. (2003) "pruning" approach (that drops terms involving 3rd and higher-order powers of the state variables from the recursion). The steady state is used for initial values; the actual length of each run was 1200 periods--the first 200 periods were discarded to ensure independence from initial conditions.

[5]Test 3 requires much longer computing times than tests 1-2, and was only computed for a few specifications (Table 3).

Row "a" of Table 1 (labeled "All specifications") reports maxima [averages] of $T_r, S_{\max}$ $[S_{mean}]$ across all 84 specifications. DJJ consider 6 combinations of different functional forms for utility/production functions. Rows b.1-b.6 shows maxima [averages] of $T_r, S_{\max}$ $[S_{mean}]$ across the specifications listed in DJJ's Problems A1, A3, A4, A5, A7 and A8, respectively. Rows c.1 and c.2 report maxima [averages] of $T_r, S_{\max}$ $[S_{mean}]$ across all specifications in which the standard deviation of productivity innovations is low $(\sigma=0.001)$, and across all variants with a high standard deviation $(\sigma=0.01)$, respectively.

For ***each*** of the 84 individual model specifications, T_r (for *all* values of *r* considered here), $S_{\max}$ and S_{mean} are lower under the quadratic approximation than under the linear approximation. The maxima of $S^Q_{\max}$ and $S^L_{\max}$ across *all* variants are $10^{-2.50}$=0.31% and $10^{-1.63}$=2.34%, respectively, while the averages of S^Q_{mean} and S^L_{mean} across *all* variants are $10^{-4.03}$=0.009% and $10^{-2.80}$=0.15%, respectively.

Accuracy is highest when the system is close to the steady state: T_r is increasing in r (distance from steady state). Across all specifications, the maxima of $T^Q_{0.01}$ and $T^L_{0.01}$ $(r=0.01)$ are $10^{-4.41}$= 0.0038% and $10^{-2.57}$=0.27%, respectively; the maxima of $T^Q_{0.3}$ and $T^L_{0.3}$ $(r=0.3)$ are $10^{-1.13}$= 7.41% and $10^{-0.59}$=25.70%, respectively.

Approximation errors are increasing in the volatility of productivity, as can be seen by comparing Rows c.1 and c.2: e.g., across all specifications with $\sigma=0.001$, the maxima of $S^Q_{\max}$ and $S^L_{\max}$ are $10^{-6.90}$=0.00001% and $10^{-4.65}$=0.002%, respectively; across all variants with $\sigma=0.01$, the maxima of $S^Q_{\max}$ and $S^L_{\max}$ are $10^{-2.50}$=0.31% and $10^{-1.63}$=2.34%, respectively.

We sorted the 84 specifications into pairs such that the specifications in a given pair only differ regarding σ and ρ (other parameters are identical across the members of that pair); there are 48 pairs of this type. Within *all* pairs, the accuracy gains $T^L_{0.01}$-$T^Q_{0.01}$, $T^L_{0.1}$-$T^Q_{0.1}$, $T^L_{0.3}$-$T^Q_{0.3}$, $S^L_{\max}-S^Q_{\max}$, $S^L_{mean}-S^Q_{mean}$, produced by the quadratic approximation (compared to the linear approx.) are greater for the specification with $(\sigma=0.01, \rho=0.95)$, than for that with $(\sigma=0.001, \rho=0.8)$. The maximum and the average of $S^L_{\max}-S^Q_{\max}$ $[S^L_{mean}-S^Q_{mean}]$ $\{T^L_{0.3}-T^Q_{0.3}\}$ across all individual specifications with $(\sigma=0.01, \rho=0.95)$ are 1.98% and 0.80% [0.73% and 0.29%] {18.05% and 4.03%}, respectively; the corresponding maximum and average accuracy gains across all cases with $(\sigma=0.001, \rho=0.8)$ are 0.002% and 0.001% [0.001% and 0.0004%] {5.96% and 1.57%}.

The preceding results show that the quadratic model solution can generate noticeably smaller approximation errors than the linear solution, especially when $\sigma=0.01$.

It seems interesting to investigate whether accuracy depends on the number of countries (*N*) or the curvature of preferences/technology. To this end, we regressed the logs of $S^L_{\max}, S^L_{mean}, S^Q_{\max}, S^Q_{mean}$ and $S^L_{\max}-S^Q_{\max}, S^L_{mean}-S^Q_{mean}$ (for all 1128 variants) on: a constant; the number of countries N; the household's intertemporal elasticity of substitution (inverse of the coefficient of relative risk aversion) γ; the capital adjustment cost parameter φ; the standard deviation of the productivity innovation σ. The results are shown in **Table 2**. Approximation errors are increasing in *N*, and in the coefficient of risk aversion, but decreasing in φ. By far, the main determinant of approximation errors and of the accuracy gains produced by the quadratic approximation is the volatility of productivity: σ is the most significant regressor; it captures between 92% and 94% of the variance of the regressands.[6]

Table 3 shows detailed results for 24 individual specifications (among the total 84 specifications described by DJJ): for each of six models (A1, A3,A4,A5,A7,A8), we pick four specifications. Cases with N=2 and N=6 are considered; in all cases, we set $\varphi = 0.5$. Panel (a) of Table 3 assumes σ=0.001, $\rho = 0.8$, which corresponds to the smallest volatility (and persistence) of productivity shocks considered by DJJ; Panel (b) assumes bigger and more persistent shocks: σ=0.01, $\rho = 0.95$ (the values of the remaining parameter values are listed in Cols. (2)-(6))

For the specifications in Table 3, the error measures $T_r, S_{\max}, S_{mean}$ are not closely linked to the number of countries. As might be expected, approximation errors are larger with bigger shocks (Panel (b)). In several cases the accuracy gain produced by the quadratic approximation is substantial. The largest accuracy gain occurs for Problem A7, with σ=0.01. There, $T_{.3}^{L}$=$10^{-0.59}$=25.70%, $S_{\max}^{L}$=$10^{-1.80}$=1.58%, $T_{.3}^{L}$=$10^{-1.13}$=7.41%, $S_{\max}^{Q}$=$10^{-2.74}$= 0.18%; thus, in this case, $T_{.3}$ and $S_{\max}$ are lower by 18.2 and 1.4 percentage points, respectively, under the quadratic approximation (compared to the linear approximation).

The simulated DM frequencies ($P_{0.05}$,$P_{0.50}$,$P_{0.95}$) are similar across the quadratic and linear approximations—however, when $\sigma = 0.01$ is assumed, the simulated frequencies favor *very* slightly the quadratic approximation.[7]

Under both approximations (and for both σ=0.001 and σ=0.01), the DM accuracy measures in Table 3 are *much* "worse" when the number of countries is large (N=6) than when N=2. E.g., for Problem A1 with σ=0.01, the quadratic approximation gives $P_{0.05}$=0.045, $P_{0.50}$=0.495, $P_{0.95}$=0.968 when N=2, compared to $P_{0.05}$=0.997 $P_{0.50}$=0.997, $P_{0.95}$=1.000 when N=6. The DM accuracy measure is sensitive to the number of instruments and to the length of the simulated series. For a sufficiently large number of instruments (and sufficiently long series), any approximate model solution fails the DM test (see discussion in DM, p.7). In Table 3, the number of instruments is $1+N+2N^2$, i.e. there are 11 instruments when N=2, and 79 instruments when N=6. Thus the number of instruments is much larger when N=6 (the largest number of instruments used by DM (1994) was 7). In experiments with less instruments and/or shorter series we detected no dependence of the DM accuracy measure on the number of countries (results available on request).

Table 4 reports the time required to produce the accuracy measures shown in Panel (a) of Table 3, using MATLAB 6.1 on a Pentium 4 PC (2.5 GHz). [8] The Columns labeled "Deriv." and "Algor." respectively show the time it takes to compute the derivatives of the model, and the time it takes to compute the solution with gensys2 (using the derivatives). For the quadratic approximation, the former is about 0.2 seconds when there are N=2 countries and about 35 seconds when N=10; gensys2 takes less than 0.1 second when N=2, and about 3 seconds when N=10. Even when the number of countries is large, gensys2 is thus very fast.

[6]Regressions on just a constant and σ yield $R^2 s$ between 0.92 and 0.95 (not reported in Table 2); with the full set of regressors, R^2 is about 0.02 higher.

[7]For σ=0.01, 22 of the 36 simulated $P_{0.05}$,$P_{0.50}$,$P_{0.95}$ frequencies are closer, up to the third decimal, to 0.05, 0.5 or 0.95, respectively, under the quadratic approximation than under the linear approximation; only 8 of the simulated frequencies are *less* close under the quadratic approximation.

[8] The inversion of a 1000×1000 matrix with i.i.d. random elements takes about 1.9 seconds on that machine. Computing times are random. The times shown in Table 4 were obtained by solving each model variant and computing each of the test statistics *once*.

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Table 1. Accuracy tests: results for sets of specifications

Linear approximation					2nd order approximation				
Test 1			Test 2		Test 1			Test 2	
$T_{.01}$	$T_{.1}$	$T_{.3}$	$S_{\max}$	S_{mean}	$T_{.01}$	$T_{.1}$	$T_{.3}$	$S_{\max}$	S_{mean}
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*
a. All specifications									
-2.57	-1.40	-0.59	-1.63	-2.80	-4.41	-2.53	-1.13	-2.50	-4.03
b.1. Problem A1									
-3.12	-1.84	-0.99	-2.30	-3.32	-5.37	-3.08	-1.71	-3.46	-4.84
b.2. Problem A3									
-2.57	-1.44	-0.64	-1.63	-2.66	-4.41	-2.53	-1.16	-2.50	-3.86
b.3. Problems A4									
-2.75	-1.60	-0.77	-1.89	-2.74	-4.77	-2.73	-1.36	-2.82	-4.00
b.4. Problems A5									
-3.12	-1.83	-0.98	-2.30	-3.29	-5.35	-3.05	-1.68	-3.45	-4.78
b.5. Problems A7									
-2.63	-1.40	-0.59	-1.70	-2.66	-4.45	-2.60	-1.13	-2.53	-3.87
b.6. Problems A8									
-2.74	-1.57	-0.75	-1.82	-2.72	-4.70	-2.67	-1.31	-2.71	-3.96
c.1. Small shocks ($\sigma = 0.001$)									
-3.95	-1.98	-1.10	-4.65	-5.31	-6.01	-3.06	-1.70	-6.90	-7.85
c.2. Big shocks ($\sigma = 0.01$)									
-2.57	-1.40	-0.59	-1.63	-2.50	-4.41	-2.53	-1.13	-2.50	-3.73

Note: This Table summarizes the results for full set of 1128 model specifications (see Row a, labeled "All specifications"), and for subsets of specifications (remaining Rows). For each (sub)set, the Columns labeled "T_r" ($r \in \{.01, .02, .05, .1, .2, .3\}$) and "$S_{\max}$" (i.e. Cols. (1)-(7) and (9)-15)) report the maximum of the error measures $T_r, S_{\max}$ across the individual model specifications (included in that set); the Cols. labeled "S_{mean}" (i.e. Cols. (8) and (16)) show averages of S_{mean} across the individual model specifications (included in the different sets of specifications). The Figures in the Table are logarithms to the base 10 ($\log_{10}$) of the maxima/ averages of $T_r, S_{\max}, S_{mean}$ (across individual specifications).

Cols. 1-8 and Cols. 9-16 of the Table pertain to a linear model solution and to the quadratic solution, respectively.

Table 2. Relating accuracy to model parameters: regression results

$\boldsymbol{log}_{10}(\boldsymbol{S}^{L,i}_{\max})$	=	-4.93	$0.005\ \boldsymbol{N}^i$	$-0.46\ \gamma^i$	$+315.8\,\boldsymbol{\sigma}^i$	$-0.02\ \boldsymbol{\varphi}^i$	$\boldsymbol{R}^2 = .96$;
		(51.25)	*(0.60)*	*(4.94)*	*(49.16)*	*(4.52)*	
$\boldsymbol{log}_{10}(\boldsymbol{S}^{Q,i}_{\max})$	=	-7.94	$0.002\,\boldsymbol{N}^i$	$-0.67\ \gamma^i$	$+455.7\,\boldsymbol{\sigma}^i$	$-0.06\,\boldsymbol{\varphi}^i$	$\boldsymbol{R}^2 = .96$;
		(49.54)	*(0.15)*	*(4.66)*	*(46.74)*	*(6.96)*	
$\boldsymbol{log}_{10}(\boldsymbol{S}^{L,i}_{\boldsymbol{mean}})$	=	-5.48	$0.030\,\boldsymbol{N}^i$	$-0.49\ \gamma^i$	$+309.3\,\boldsymbol{\sigma}^i$	$-0.02\,\boldsymbol{\varphi}^i$	$\boldsymbol{R}^2 = .96$;
		(56.05)	*(3.38)*	*(5.08)*	*(47.28)*	*(4.40)*	
$\boldsymbol{log}_{10}(\boldsymbol{S}^{Q,i}_{\boldsymbol{mean}})$	=	-7.89	$0.030\,\boldsymbol{N}^i$	$-0.74\ \gamma^i$	$+443.9\,\boldsymbol{\sigma}^i$	$-0.06\,\boldsymbol{\varphi}^i$	$\boldsymbol{R}^2 = .96$;
		(53.69)	*(2.19)*	*(5.14)*	*(45.13)*	*(6.76)*	
$\boldsymbol{log}_{10}(\boldsymbol{S}^{L,i}_{\max} - \boldsymbol{S}^{Q,i}_{\max})$	=	-4.94	$0.005\,\boldsymbol{N}^i$	$-0.46\ \gamma^i$	$+312.5\,\boldsymbol{\sigma}^i$	$-0.02\,\boldsymbol{\varphi}^i$	$\boldsymbol{R}^2 = .96$;
		(52.32)	*(0.57)*	*(4.96)*	*(49.61)*	*(4.41)*	
$\boldsymbol{log}_{10}(\boldsymbol{S}^{L,i}_{\boldsymbol{mean}} - \boldsymbol{S}^{Q,i}_{\boldsymbol{mean}})$	=	-5.48	$0.030\ \boldsymbol{N}^i$	$-0.48\ \gamma^i$	$+307.2\,\boldsymbol{\sigma}^i$	$-0.02\,\boldsymbol{\varphi}^i$	$\boldsymbol{R}^2 = .96$.
		(56.81)	*(3.40)*	*(5.10)*	*(47.55)*	*(4.33)*	

Note: The Table shows regressions of the logged accuracy measures for the 1128 model variants on a constant and on parameters used in these variants. $\boldsymbol{S}^{L,i}_{\max}$, $\boldsymbol{S}^{Q,i}_{\max}$, $\boldsymbol{S}^{L,i}_{\boldsymbol{mean}}$, $\boldsymbol{S}^{Q,i}_{\boldsymbol{mean}}$ and $\boldsymbol{N}^i$, γ^i, $\boldsymbol{\varphi}^i$, $\boldsymbol{\sigma}^i$ and $\boldsymbol{\rho}^i$ are accuracy measures and parameters for variant *i*. (For model variants in which γ differs across countries, the regression uses the mean of γ across countries.) $\boldsymbol{S}^{L,i}_{\max}$ and $\boldsymbol{S}^{L,i}_{\boldsymbol{mean}}$ pertain to the linear approximation, while $\boldsymbol{S}^{Q,i}_{\max}$ and $\boldsymbol{S}^{Q,i}_{\boldsymbol{mean}}$ pertain to the quadratic approximation.

The figures in parentheses below the regression coefficients are absolute values of t-statistics.

Table 3. Accuracy tests: results for selected specifications

Mo= del	N	γ	η	χ	μ	Linear approximation: Test 1 $T_{.01}$	Test 1 $T_{.3}$	Test 2 $S_{\max}$	Test 2 S_{mean}	Test 3 $P_{.05}$	Test 3 $P_{.5}$	Test 3 $P_{.95}$	2nd order approximation: Test 1 $T_{.01}$	Test 1 $T_{.3}$	Test 2 $S_{\max}$	Test 2 S_{mean}	Test 3 $P_{.05}$	Test 3 $P_{.5}$	Test 3 $P_{.95}$
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*	*(16)*	*(17)*	*(19)*	*(20)*	*(21)*
***(a) Small shocks:* σ = 0.001**																			
A1	2	1	--	--	--	-4.18	-1.33	-5.36	-6.01	.04	.49	.96	-6.33	-2.01	-7.61	-8.32	.04	.49	.96
A1	6	1	--	--	--	-4.70	-1.85	-5.49	-5.84	.99	.99	1.00	-7.01	-2.66	-7.65	-8.22	.99	.99	1.00
A3	2	.25	--	--	--	-4.13	-1.28	-4.78	-5.27	.04	.46	.95	-6.15	-1.85	-7.22	-7.75	.04	.45	.95
A3	6	.25	--	--	--	-4.69	-1.83	-4.65	-5.04	.00	.23	.89	-6.83	-2.52	-7.17	-7.56	.00	.23	.88
A4	2	.25	--	.83	-.20	-4.09	-1.24	-4.88	-5.34	.04	.46	.95	-6.11	-1.81	-7.22	-7.80	.04	.45	.95
A4	6	.25	--	.83	-.20	-4.63	-1.78	-4.73	-5.12	.00	.24	.89	-6.81	-2.49	-7.26	-7.63	.00	.24	.89
A5	2	.25/1	--	--	--	-4.17	-1.31	-5.27	-5.90	.04	.47	.95	-6.28	-1.96	-7.51	-8.26	.04	.47	.95
A5	6	.25/1	--	--	--	-4.69	-1.81	-5.43	-5.82	.99	.99	1.00	-6.95	-2.61	-7.65	-8.18	.99	.99	1.00
A7	2	.25/1	--	--	--	-3.95	-1.10	-4.89	-5.28	.03	.47	.96	-6.07	-1.71	-6.93	-7.75	.03	.48	.95
A7	6	.25/1	--	--	--	-4.63	-1.76	-4.73	-5.10	.00	.23	.88	-6.70	-2.35	-7.07	-7.61	.00	.23	.88
A8	2	.2/.4	--	.75/.9	−.3/.3	-3.98	-1.13	-4.87	-5.31	.04	.46	.95	-6.00	-1.70	-7.07	-7.75	.04	.45	.95
A8	6	.2/.4	--	.75/.9	−.3/.3	-4.63	-1.76	-4.73	-5.12	.00	.22	.88	-6.85	-2.55	-7.18	-7.60	.00	.23	.88
***(b) Big shocks*: σ = 0.01**																			
A1	2	1	--	----		-3.26	-0.99	-2.34	-3.02	.03	.43	.94	-5.40	-1.71	-3.66	-4.44	.04	.45	.94
A1	6	1	--	--	--	-3.15	-1.51	-2.35	-2.80	.00	.06	.63	-5.41	-2.19	-3.69	-4.23	.00	.06	.63
A3	2	.25	--	--	--	-2.88	-0.74	-1.88	-2.45	.03	.39	.93	-4.95	-1.32	-2.93	-3.64	.03	.42	.96
A3	6	.25	--	--	--	-2.73	-1.14	-1.81	-2.18	.00	.04	.55	-4.82	-1.79	-2.85	-3.35	.00	.05	.57
A4	2	.25	--	.83	-.20	-2.92	-0.77	-1.94	-2.52	.03	.39	.94	-4.99	-1.36	-3.06	-3.71	.03	.41	.96
A4	6	.25	--	.83	-.20	-2.72	-1.19	-1.89	-2.25	.00	.05	.60	-4.87	-1.81	-2.91	-3.45	.00	.06	.61
A5	2	.25/1	--	--	--	-3.26	-0.98	-2.32	-2.92	.04	.41	.94	-5.43	-1.68	-3.71	-4.36	.04	.42	.95
A5	6	.25/1	--	--	--	-3.15	-1.50	-2.32	-2.79	.00	.06	.64	-5.40	-2.17	-3.57	-4.18	.00	.06	.65
A7	2	.25/1	--	--	--	-2.80	-0.59	-1.80	-2.37	.03	.39	.91	-4.45	-1.13	-2.74	-3.55	.02	.43	.94
A7	6	.25/1	--	--	--	-2.66	-1.07	-1.81	-2.15	.00	.04	.54	-4.55	-1.70	-2.71	-3.30	.00	.04	.55
A8	2	.2/.4	--	.75/.9	−.3/.3	-2.91	-0.75	-1.95	-2.47	.03	.41	.92	-4.79	-1.31	-3.01	-3.72	.03	.41	.95
A8	6	.2/.4	--	.75/.9	−.3/.3	-2.76	-1.16	-1.83	-2.23	.00	.04	.56	-4.79	-1.82	-2.82	-3.41	.00	.05	.58

Note: Cols. 1-6 list the model, N, and the preference and technology parameters γ,η,χ,μ. When a single number is reported in Cols. (3)-(6), this indicates a parameter value that is common to all countries; an entry x/y for a given parameter ξ indicates that country $j=1,..,N$ has parameter $\xi_j = x+((j-1)/(N-1))\cdot(y-x)$. The figures shown for Tests 1 and 2 are logs of the error measures $(\log_{10}(T_r), \log_{10}(S_{\max}), \log_{10}(S_{mean}))$.

Table 4. Computing times for selected models

Mo= del	N	γ	η	χ	μ	Linear approximation: Deriv.	Linear approximation: Algor.	Linear approximation: Test 1	Linear approximation: Test 2	Linear approximation: Test 3	2nd order approximation: Deriv.	2nd order approximation: Algor.	2nd order approximation: Test 1	2nd order approximation: Test 2	2nd order approximation: Test 3
(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*	*(16)*
A1	2	.25	--	--	--	0.01	0.06	12	12	161	0.09	0.09	13	15	364
A1	10	.25	--	--	--	0.03	0.39	56	50	1838	34.95	3.36	89	96	3586
A3	2	.25	--	--	--	0.02	0.03	15	12	83	0.14	0.04	14	15	334
A3	6	.25	--	--	--	0.03	0.09	31	31	304	3.67	0.29	38	40	745
A4	2	.25	--	.83	-.20	0.02	0.04	13	13	102	0.26	0.05	14	15	358
A4	6	.25	--	.83	-.20	0.05	0.10	32	32	351	6.87	0.30	39	41	838

Time (seconds)

Note: Columns labeled **"Deriv.", "Algor."**: computing time of derivatives of model, and computing time of the solution, respectively. Cols. Labeled **"Test 1":** computing time for accuracy statistic T_r (for r=0.01). Cols. Labeled **"Test 2" ["Test 3"]**: computing time for accuracy statistics S_{max}, S_{mean} [$P_{.05}, P_{.5}, P_{.95}$].