Perturbation Analysis

of the Incomplete Markets Case

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- Why Perturbation? Ans.: Speed. (e.g. September 18th FOMC meeting)

- Challenges of Applying a Perturbation Method

  - #1: Continuum of Agents

  - #2: (Hard) Inequality Constraints

  - #3: Discrete Shocks

- How do K^3 meet these three challenges?

- Challenge #1: Continuum of Agents
  - First Method: One Infinitesimal Agent and the Rest of the World
    * The agent faces factor prices given by ROW.
    * ROW as a representative agent?
      · Only aggregate shocks.
      · Neglecting the dynamics of wealth distribution.
    * Legitimate under first-order approximation.
    * An example on the next page (K^3 on problem C).
  - Second Method: ???

- Bond Economy *without Inequality Constraints*: (e.g. KKL, 2003)

$$\max \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} \text{ subject to } c_t^i + q_t b_t^i = y_t^i a_t + b_{t-1}^i.$$

- First-Order Condition: $q_t \left(c_t^i\right)^{-\gamma} = \beta \mathsf{E}_t \left(c_{t+1}^i\right)^{-\gamma}$

- Playing around with the *deterministic* case, we have

$$c_{t+1}^i = \left(\frac{\beta}{q_t}\right)^{\frac{1}{\gamma}} c_t^i$$

- Aggregating,

$$\sum_i c_{t+1}^i = \left(\frac{\beta}{q_t}\right)^{\frac{1}{\gamma}} \sum_i c_t^i$$

- "approximate" aggregation (i.e. E's) *&* "approximate" representative agent

- (Cont'd) Challenge #1: Continuum of Agents
  - First Method: One Infinitesimal Agent and the Rest of the World
    - The agent faces factor prices given by ROW.
    - ROW as a representative agent?
      - Only aggregate shocks.
      - Neglecting the dynamics of wealth distribution.
    - Legitimate under first-order approximation.
    - An example on the next page (K^3 on problem C).
  - Second Method: as many as possible (e.g. one hundred countries)

- Challenge #2: (Hard) Inequality Constraints
  - variable transformation (e.g. level or log)

$$\hat{b}_t^i = \bar{b} \log \left(\frac{b_t^i + \bar{b}}{\bar{b}} \right)$$

  - hard or soft?
    * barrier method (e.g. zero lower bound for nominal interest rate)
    * penalty method (e.g. lower bound for asset holdings)
    * an hybrid one?

- Challenge #3: Discrete Shocks
  - intuition: matching some selective moments
  - details (to be continued)

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Kim-Kim-Kollmann
Solution to Problem B
Quantitative Results

2 ECONOMIES:

1) **economy with infinitesimal agent facing prices given by rest of world with representative agent subjected to aggregate productivity shocks (as in KKK paper on Problem C).**

2) **economy with finite number of agents, N; using quadratic approx can go to N=100**

Forcing variables: continuous processes chosen to match discrete state model

<u>Productivity:</u> AR(1)

<u>Individual employment status:</u>
AR(1) with state contingent coeffs.; match conditional expectation and variance of employment at t+1, given employment at t, and productivity at t & t+1

<u>Economy without aggregate risk</u>

$$d\varepsilon_{t+1}^{i} = 0.35 d\varepsilon_{t}^{i} + (0.22 - 0.28 d\varepsilon_{t}^{i})\eta_{t+1}^{i}.$$

$d\varepsilon_{t}^{i} \equiv \varepsilon_{t}^{i} - \bar{\varepsilon}$ (deviation from steady state).

η_{t+1}^{i}: white noise disturbance, $\int \eta_{t+1}^{i} di = 0$. $std(\eta_{t+1}^{i}) = 1$.

<u>Economy with aggregate risk</u>

Aggregate productivity:

$$da_{t+1} = 0.75 da_{t} + 0.0066\eta_{t+1}^{a}, \quad std(\eta_{t+1}^{a}) = 1.$$

Aggregate employment, $\bar{e}_t$: $\quad d\bar{e}_t = 3da_t$.

Employment of agent i:

$$d\varepsilon_{t+1}^{i} - d\overline{e_{t+1}} = [0.42 - 17.46 da_{t+1} + 4.84 da_t](d\varepsilon_t^i - d\bar{e}_t) + \quad \ldots..$$
$$[0.20 - 2.40 da_{t+1} + 0.10 da_t - 0.267 d\varepsilon_t^i]\,\eta_{t+1}^{i}$$

How to handle $k_{t+1}^{j} \geq 0$ constraint:

1) take expansion in $\ln(k_{t+1}^{j})$. Problems: imposes $k_{t+1}^{j} > 0$ (may not be true); big approximation errors for budget constraint when k_t^j is close to zero.

2) expansion in k_{t+1}^{j}; when k_{t+1}^{*j}, $c_t^{*,j}$ generated by linear/quadratic approximation is such that $k_{t+1}^{*j} < 0$, then replace by:
$k_{t+1}^{j} = 0, \;\; c_t^j = \; \mathrm{c}_{t+1}^{*j} \; + k_{t+1}^{*j}$**, and use** $k_{t+1}^{j} = 0$
to determine date t+1 choices.

Policy function from lin./quadr. approx:

$$k_{t+1}^{*j} = F(\varepsilon_t^j, k_t^j, K_t, a_t), \; c_t^{*j} = G(\varepsilon_t^j, k_t^j, K_t, a_t)$$

If $k_{t+1}^{*j} < 0$, then set $k_{t+2}^{*j} = F(\varepsilon_{t+1}^{j}, 0, K_{t+1}, a_{t+1})$
$c_{t+1}^{*j} = G(\varepsilon_{t+1}^{j}, 0, K_{t+1}, a_{t+1})$.

RESULTS

Model without aggregate risk

Capital choices, as function of money at hand: Very similar across employed and unemployed.

Broadly similar across linear and quadratic approx.,

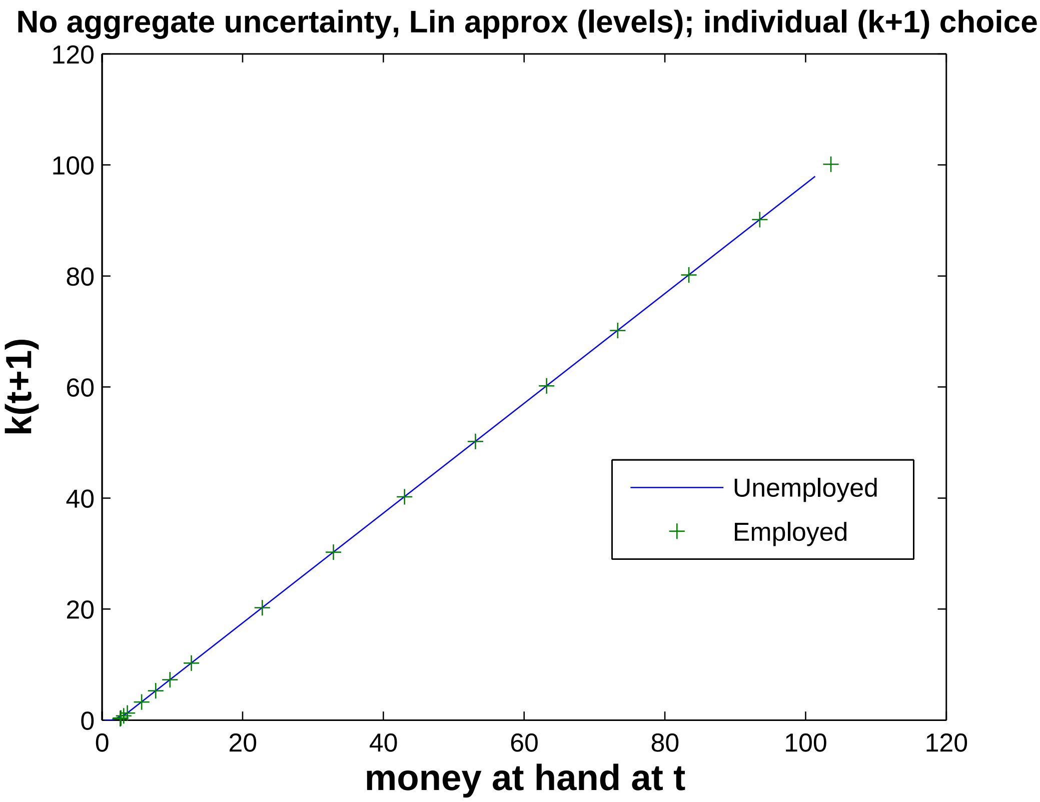
Linear approx. slightly more accurate than quadratic approx for small and big k.

LINEAR APPROXIMATION:

Money at hand & k_t^i at which $k_{t+1}^i \geq 0$ just binds

	$\varepsilon_t^i = 0$		$\varepsilon_t^i = 0.5$		$\varepsilon_t^i = 0.75$	
	Zero penalty	**Optimal penalty**	**Zero penalty**	**Optimal penalty**	**Zero penalty**	**Optimal penalty**
Money at hand	**2.34**	**2.12**	**2.35**	**2.05**	**2.35**	**2.06**
k_t^i	**1.97**	**1.75**	**0.97**	**0.57**	**0.32**	**0.03**
Penalty coeff.	**0.00**	**0.016**	**0.00**	**0.016**	**0.00**	**0.016**

Optimal penalty: minimizes Euler & budget error across $0 \leq k_t^i < 100$**,** $\varepsilon_t^i = 0, 1.$


No aggregate uncertainty, Lin approx (levels); individual (k+1) choice
k(t+1)
money at hand at t
0
20
40
60
80
100
120
Unemployed
Employed

APPROXIMATION ERRORS

Euler equation error: $\{1-E_t\beta\frac{U^i_{C,t+1}}{U^i_{C,t}}[R_{t+1}+1-\delta]\}k^i_{t+1}$

Budget equation error: $E_t\left|[-(c^i_{t+1}+k^i_{t+2})+money\ at\ hand^i_{t+1}]/(money\ at\ hand^i_{t+1})\right|$

<u>NO</u> BORROWING CONSTRAINT, Linear approximation, zero penalty coeff:

Unemployed agent			Employed agent			
Euler	Budget	k^i_{t+1}	Euler	Budget	k^i_{t+1}	k^i_t
0.0000	0.0000	-1.8702	0.0000	0.0000	0.3189	0.1
0.0000	0.0000	-0.4702	0.0000	0.0000	1.7189	1.5
0.0000	0.0000	0.0298	0.0000	0.0000	2.2189	2.0
0.0000	0.0000	2.0298	0.0000	0.0000	4.2189	4.0
0.0000	0.0000	18.0298	0.0000	0.0000	20.2189	20.0
0.0000	-0.0000	78.0298	0.0000	-0.0000	80.2189	80.0

ERRORS WITH BORROWING CONSTRAINT:

1a) Linear approximation, $k^{j}_{t+1} \geq 0$ adjustment, zero penalty coeff:

Unemployed agent			Employed agent			
Euler	Budget	k^{i}_{t+1}	Euler	Budget	k^{i}_{t+1}	k^{i}_{t}
0	0.0000	0	0.0383	0.0000	0.3189	0.1
0	0.0000	0	0.0000	0.0000	1.7189	1.5
1.5302	0.0000	0.0298	0.0000	0.0000	2.2189	2.0
0.0026	0.0000	2.0298	0.0000	0.0000	4.2189	4.0
0.0000	0.0000	18.0298	0.0000	0.0000	20.2189	20.0
0.0000	0.0000	78.0298	0.0000	0.0000	80.2189	80.0

1b) Linear approximation, $k^{j}_{t+1} \geq 0$ adjustment, optimal penalty coeff: 0.0161

Unemployed agent			Employed agent			
Euler	Budget	k^{i}_{t+1}	Euler	Budget	k^{i}_{t+1}	k^{i}_{t}
0	0.0000	0	0.0041	0.0000	0.5217	0.1
0	0.0000	0	-0.0013	0.0000	1.9142	1.5
0.7138	0.0000	0.2485	-0.0013	0.0000	2.4114	2.0
-0.0013	0.0000	2.2377	-0.0012	0.0000	4.4006	4.0
-0.0006	0.0000	18.1509	-0.0006	0.0000	20.3138	20.0
0.0010	0.0000	77.8256	0.0011	0.0000	79.9885	80.0

2) Quadratic approximation, $k^{j}_{t+1} \geq 0$ adjustment, zero penalty coefficient

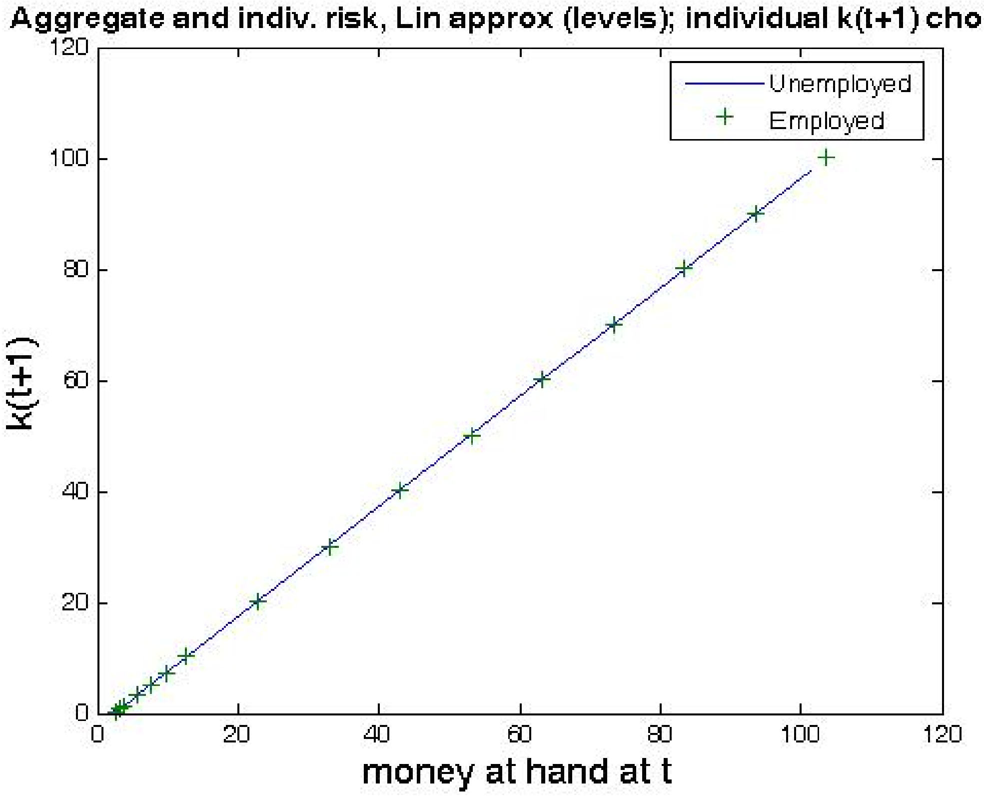
Unemployed agent			Employed agent			
Euler	Budget	k^{i}_{t+1}	Euler	Budget	k^{i}_{t+1}	k^{i}_{t}
0	0.0000	0	0.0259	0.0000	0.3238	0.1
0	0.0000	0	-0.0000	0.0000	1.7238	1.5
-0.9436	0.0000	0.0347	-0.0000	0.0000	2.2238	2.0
0.1102	0.0000	2.0347	-0.0000	0.0000	4.2238	4.0
0.0001	0.0000	18.0347	-0.0000	0.0000	20.2238	20.0
0.0000	0.0000	78.0347	-0.0000	0.0000	80.2238	80.0

3) Log-linear approximation, zero penalty coefficient (optimal)

Unemployed agent			Employed agent			
Euler	Budget	k_{t+1}^i	Euler	Budget	k_{t+1}^i	k_t^i
0.0000	0.4501	0.0949	0.0000	0.4737	0.1006	0.1
0.0000	0.2444	1.4242	0.0000	0.1824	1.5087	1.5
0.0000	0.2114	1.8989	0.0000	0.1506	2.0116	2.0
0.0000	0.1361	3.7978	0.0000	0.0877	4.0231	4.0
0.0000	0.0249	18.9892	0.0000	0.0094	20.1156	20.0
0.0000	0.0005	37.9784	0.0000	0.0005	40.2312	40.0
0.0000	0.0150	75.9567	0.0000	0.0043	80.4623	80.0

Model with aggregate risk

Capital choices for employed and unemployed agent, as function of money at hand (aggregate capital stock and aggregate productivity set at steady state values) (penalty coeff. set at value that minimizes error: 0.003)



Model with aggregate uncertainty

Simulate 200 agents for 2500 periods,
discard first 1000 periods

Linear approximation, with $k_{t+1}^{j} \geq 0$ adjustment,
optimal penalty coeff used (0.003)

RISK SHARING

	KKK	AAD
corr(ci,C)	0.115	0.25
corr(ci,gdp)	0.169	0.18
corr(ci,K)	0.088	
corr(ci,ki)	0.965	0.86
corr(ci,Income i)	0.968	0.22 (?)
std(ci)	0.163	0.17
std(K)	14.236	
autocorr(ci,1)	0.993	0.97
autocorr(ci,2)	0.987	0.95
autocorr(ci,3)	0.982	0.92
autocorr(K,1)	0.999	0.99
autocorr(K,2)	0.998	0.99
autocorr(K,3)	0.996	0.99

WEALTH DISTRIBUTION (ki)	KKK
fraction ki=0	0.064 %
fraction ki=0, when a>1	0.047 %
fraction ki=0, when a<1	0.079 %
mean 5th percentile	14.056
mean 5th percent., when a>1	14.523
mean 5th percent., when a<1	13.626
mean 10th percentile	19.900
mean 10th percent., when a>1	20.312
mean 10th percent., when a<1	19.521