Comparing numerical solutions of models with heterogeneous agents (Models B): a grid-based Euler equation algorithm

Lilia Maliar, Serguei Maliar and Fernando Valli [*,†]

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Abstract

This paper describes a numerical method for solving Model B (continuum of agents and saving through capital) of the JEDC project. At the individiual level, we use an Euler equation algorithm which computes a policy function on a grid of prespecified points. At the aggregate level, we describe the economy's state by the first and the second moments of the wealth distribution, and we solve for the Aggregate Law of Motion (ALM) by Monte Carlo simulation, as in Krusell and Smith (1998). Also, we propose a simple and fast bisection and updating method for finding the ALM, which in some cases, can be a useful alternative to Krusell and Smith's (1998) updating.

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[†] Departamento de Fundamentos del Análisis Económico, Universidad de Alicante, Campus San Vicente del Raspeig, Ap. Correos 99, 03080 Alicante, Spain. E-mails: *maliarl@merlin.fae.ua.es* (Lilia Maliar), *maliars@merlin.fae.ua.es* (Serguei Maliar), *valli@merlin.fae.ua.es* (Fernando Valli).

1 Introduction

This paper describes a numerical method for solving Model B (continuum of agents and saving through capital) of the JEDC project. At the individual level, we use an Euler equation algorithm which computes a policy function on a grid of prespecified points. At the aggregate level, we describe the economy's state by the first and the second moments of the wealth distribution, and we solve for the Aggregate Law of Motion (ALM) by Monte Carlo simulation, as in Krusell and Smith (1998). Also, we propose a simple and fast bisection and updating method for finding the ALM, which in some cases, can be a useful alternative to Krusell and Smith's (1998) updating.

The plan of the paper is as follows: Section 2 describes the model. Section 3 presents the algorithm. Section 4 describes the calibration and the algorithm's parameters. Section 4 discusses the results, and finally, Section 5 concludes.

2 Model B

We study a variant of Krusell and Smith's (1998) model. The economy is composed of a set of heterogeneous agents and a representative firm. Agent i solves the following problem

$$\max_{\left\{c_t^i, k_{t+1}^i\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^i\right)^{1-\gamma} - 1}{1-\gamma} \tag{1}$$

subject to

$$c_t^i + k_{t+1}^i = r_t k_t^i + \left(1 - \tau_t\right) w_t \bar{l} \varepsilon_t^i + \mu w_t \left(1 - \varepsilon_t^i\right) + \left(1 - \delta\right) k_t^i, \tag{2}$$

$$k_{t+1}^i \geq 0, \tag{3}$$

where $k_0^i > 0$ is given. Here, c_t^i is consumption; k_t^i is capital; ε_t^i is idiosyncratic shock that determines the employment status, with $\varepsilon_t^i = 1$ and $\varepsilon_t^i = 0$ corresponding to the employed and the unemployed states, respectively; $\beta \in (0, 1)$ is the discount factor; $\delta \in (0, 1]$ is the depreciation rate of capital; $\gamma > 0$ is the utility-function parameter; $\bar{l}$ is the time endowment; r_t, w_t, μw_t and τ_t are the interest rate, wage, unemployment benefit and labor-income tax rate, respectively, and finally, (3) is a restriction on borrowing.

The profit-maximizing conditions of the representative firm are

$$r_t = \alpha a_t \left(\frac{K_t}{\bar{l} L_t} \right)^{\alpha - 1}, \tag{4}$$

$$w_t = (1 - \alpha) \, a_t \left(\frac{K_t}{\bar{l} L_t} \right)^{\alpha}, \tag{5}$$

where K_t and L_t are the aggregate capital and labor, respectively; a_t is the aggregate productivity shock that can take two values $1 - \Delta^a$ and $1 + \Delta^a$; and $\alpha \in (0, 1)$ is the share of capital in production.

The government redistribute funds from employed to unemployed agents by setting the tax rate

$$\tau_t = \frac{\mu u_t}{\bar{l} L_t} \tag{6}$$

where $u_t = 1 - L_t$ is an unemployment rate.

3 The algorithm

In this section, we first discuss how to find a solution to the individual problem by using an Euler equation method described in Maliar and Maliar (2006). We next describe a bisection procedure for solving the model without aggregate uncertainty. We then outline Krusell and Smith's (1998) updating procedure for computing the ALM in the model with aggregate uncertainty. We finally propose a bisection and updating method for finding the ALM that can be a useful alternative to Krusell and Smith's (1998) updating.

3.1 The individual problem

In our economy, each agent solves the problem $(1) - (3)$, which is a variant of the problem with an occasionally binding inequality constraint. The Euler equation associated with this problem can be written as

$$C(k, \varepsilon, m, a)^{-\gamma} - h(k, \varepsilon, m, a)$$
$$= \delta E \left\{ C(k', \varepsilon', m', a')^{-\gamma} \left[1 - \delta + \alpha a' \left(\frac{K'}{\bar{l} L'} \right)^{\alpha - 1} \right] \right\}, \tag{7}$$

where variables without and with primes refer to the current and future periods, respectively; $C(k, \varepsilon, m, a)$ is a time-invariant consumption function; $h(k, \varepsilon, m, a)$ is the Lagrange multiplier associated with the borrowing constraint (3); and m is a set of statistics characterizing the wealth distribution. (We omit the individual superscripts for the sake of notational convenience).

The corresponding set of Kuhn-Tucker conditions is given by

$$h(k, \varepsilon, m, a) \geq 0, \tag{8}$$

$$A(k, \varepsilon, m, a) \geq 0, \qquad h(k, \varepsilon, m, a)\, A(k, \varepsilon, m, a) = 0, \tag{9}$$

where $A(k, \varepsilon, m, a)$ is a time-invariant asset function

$$\begin{aligned} A(k, \varepsilon, m, a) &\equiv k' \\ &= rk + \left(1 - \frac{\mu u}{\bar{l} L}\right) w \bar{l} \varepsilon + w(1-\varepsilon) + (1-\delta) k - C(k, \varepsilon, m, a). \end{aligned} \tag{10}$$

Therefore, we are to solve for $C(k, \varepsilon, m, a)$, $A(k, \varepsilon, m, a)$ and $h(k, \varepsilon, m, a)$ satisfying the Euler equation (7), Kuhn-Tucker conditions (8) and (9) and budget constraint (10).

Our solution method is similar to the parameterized expectations algorithm used in den Haan and Marcet (1990), Christiano and Fisher (2000), Maliar and Maliar (2003), and Algan and Allais (2003), however, unlike those papers, we parameterize the asset function and not the expectation term in the Euler equation. Furthermore, we do not use a polynomial parameterization, but compute the solution on a grid of prespecified points. The grid for capital (asset) holdings consists of a number of points in the range $[k_{\min}, k_{\max}]$. To evaluate the asset function outside the grid, we use a polynomial interpolation.

By substituting budget constraint (10) in the Euler equation (7), we obtain

$$\begin{aligned} c = \Bigg\{ & h(\cdot) + \\ & \delta E \left[\frac{1 - \delta + r'}{\left(\left(1 - \frac{\mu u'}{\bar{l} L'}\right) \bar{l} w' \varepsilon' + w'(1-\varepsilon') + (1-\delta+r') A(\cdot) - A(A(\cdot)) \right)^{\gamma}} \right] \Bigg\}^{-1/\gamma} \end{aligned} \tag{11}$$

where $h(\cdot) \equiv h(k, \varepsilon, m, a)$, $A(\cdot) \equiv A(k, \varepsilon, m, a)$ and $A(A(\cdot)) \equiv A(A(k, \varepsilon, m, a))$.

Consequently, we implement the following iterative procedure:

- *Step* 1. Fix some initial asset function, $A(k, \varepsilon, m, a)$, on the grid. We set the initial asset function to 0.9 times a grid value of the individual capital, $A(k, \varepsilon, m, a) = 0.9k$ for all k, ε, m, a.

- *Step* 2. Use the assumed asset function $A(k, \varepsilon, m, a)$ to calculate consumption in (11) in each point on the grid by setting the Lagrange multiplier equal to zero, i.e., $h(k, \varepsilon, m, a) = 0$ for all k, ε, m, a. Compute the new asset function, $\widetilde{A}(k, \varepsilon, m, a)$, from (10). For each point of the grid, in which $\widetilde{A}(k, \varepsilon, m, a)$ does not belong to $[k_{\min}, k_{\max}]$, set $\widetilde{A}(k, \varepsilon, m, a)$ at the corresponding boundary value.

- *Step* 3. Compute the asset function for next iteration $\widetilde{\widetilde{A}}(k, \varepsilon, m, a)$ by using updating:

$$\widetilde{\widetilde{A}}(k, \varepsilon, m, a) = \eta_A \widetilde{A}(k, \varepsilon, m, a) + (1 - \eta_A) A(k, \varepsilon, m, a), \quad \eta_A \in (0, 1]. \tag{12}$$

Iterate on *Steps* $2-3$ until $\widetilde{\widetilde{A}}(k, \varepsilon, m, a) = A(k, \varepsilon, m, a)$ with a given degree of precision, 10^{-10}, according to the least-square norm.

Note that by construction, the obtained solution satisfies the Euler equation (7), the Kuhn-Tucker conditions in (9) and budget constraint (10). We are left to check that our solution satisfies the remaining Kuhn-Tucker condition (8), i.e., that the Lagrange multiplier is non-negative whenever the borrowing constraint (3) binds. Notice that under $\gamma > 0$, the term $\{h(k, \varepsilon, m, a) + ...\}^{-1/\gamma}$ in (11) is decreasing in the value of $h(k, \varepsilon, m, a)$. Since when the unconstrained solution (obtained under $h(k, \varepsilon, m, a) = 0$) violated the borrowing constraint (3), we set the asset holdings in the left side of (3) at the borrowing limit, we should increase the Lagrange multiplier in the right side of (11) in order to preserve the equality sign. Hence, our method insures that the Lagrange multiplier is always non-negative.

3.2 No aggregate uncertainty: a bisection method

We first consider a model without aggregate uncertainty. In order to solve such a model, we are to find an interest rate, which is consistent with the stochastic steady state. This can be done by a standard bisection method computing a one-dimensional fixed point, as described in Huggett (1993) and Aiyagari (1994).

Suppose that we know two values, $\underline{r}$ and $\overline{r}$, on the opposite sides of the steady state interest rate r^*, such that $\underline{r} < r^* < \overline{r}$. Then, we can find the equilibrium interest rate as follows:

- *Step I*. Define $r^{bis} = (\overline{r} + \underline{r})/2$. Fix the initial wealth distribution across N heterogeneous agents. For each agent, generate and fix time series for the idiosyncratic shocks of length T. We discard the first T_1 periods in order to eliminate the effect of initial conditions.
- *Step II*. Given r^{bis}, compute a solution to the individual problem as described in Section 3.1.
- *Step III*. Use r^{bis} and the individual decision rules computed in *Step II* to simulate the economy T periods forward by explicitly solving for the asset holdings of each agent $i = 1, ..., N$.
- *Step IV*. Use the time series for the individual asset holdings calculated in *Step III* in order to mean of the wealth distribution.
- *Step V*. Deduce the corresponding interest rate $\widetilde{r}$.
- *Step VI*. If $\widetilde{r} > r^{bis}$, let $\overline{r} = r^{bis}$; otherwise, let $\underline{r} = r^{bis}$.

Iterate on *Steps* $II - VI$ until $\widetilde{r} = r^{bis}$ with a given degree of precision.

The bisection method works very fast and guarantees the convergence: typically it requires about 20 iterations for a very accurate approximation.

3.3 Aggregate uncertainty

We describe two methods for finding the ALM in the model with aggregate uncertainty: one is Krusell and Smith's (1998) updating and the other is a combination of bisection and updating.

3.3.1 An updating method

To solve for the ALM in the model with aggregate uncertainty, we can use an updating method proposed in Krusell and Smith (1998). Choose a set of statistics $m = \{m_1, ..., m_M\}$ for describing the wealth distribution and parameterize the ALM of m by a flexible functional form of the current m, i.e.,

$$m' = f(b, m), \tag{13}$$

where $b = (b_1, ..., b_S)$ is a vector of coefficients to be computed. In our experiments, we consider m consisting of either the first moment (mean) or the first and second moments (mean and variance) of the wealth distribution. We consider four values for the mean and the variance, which are uniformly distributed on the appropriate intervals.

We solve for the ALM by using the following updating procedure:

- *Step I*. Fix some initial vector b. Generate and fix time series of length T for the aggregate shocks. Fix the initial wealth distribution across N heterogeneous agents. For each agent, generate and fix time series for the idiosyncratic shocks of length T. We discard the first T_1 periods when re-estimating (13) in order to eliminate the effect of initial conditions.

- *Step II*. Given b and ALM (13), compute a solution to the individual problem as described in Section 3.1.

- *Step III*. Use ALM (13) and the individual decision rules computed in *Step II* to simulate the economy T periods forward by explicitly solving for the asset holdings of each agent $i = 1, ..., N$ and by calculating the set of statistics m for each $t = 1, ..., T$.

- *Step IV*. Use the time series for the statistics m calculated in *Step III* in order to re-estimate the ALM coefficients in (13).

- *Step V*. Call the resulting vector of the ALM coefficients in (13) by $\widetilde{b}$.

- *Step VI*. Compute the ALM coefficients for next iteration by using updating:

$$\overset{\approx}{b} = \eta_b \widetilde{b} + (1 - \eta_b) b, \qquad \eta_b \in (0, 1]. \tag{14}$$

Iterate on *Steps II − VI* until $\overset{\approx}{b} = b$ with a given degree of precision.

3.3.2 A bisection and updating method

We now propose a bisection and updating method that can be used as a fast alternative to Krusell and Smith's (1998) updating. The standard one-dimensional method is fast and efficient but unfortunately, cannot be directly extended to more than one dimension because there is no direct way to implement a bisection on vectors. So, for multi-dimensional case, we propose a method that combines both bisection and updating, namely, we perform a bisection with respect to one of the coefficients, and we perform an updating of the rest of the coefficients. To be precise, let $b = (b_1, ..., b_S)$, and without a loss of generality, let b_S be the bisection coefficient. We replace *Step V* of the algorithm of Section 3.3.1 by the following steps.

- *Step* Va'. Call the resulting vector of the ALM coefficients in (13) by $\widetilde{b} = \left(\widetilde{b}_1, ..., \widetilde{b}_S\right)$. If $\widetilde{b}_S > b_S$, let $\overline{b} = \widetilde{b}$ and $\underline{b} = b$; otherwise, let $\overline{b} = b$ and $\underline{b} = \widetilde{b}$.
- *Step* Vb'. Let $b^{bis} = \left(\overline{b} + \underline{b}\right)/2$. Perform *Steps* $II - IV$ of the algorithm of Section 3.2 for the ALM vector b^{bis}, and get the new vector $\widetilde{b}$. If $\widetilde{b}_S > b_S^{bis}$ let $\overline{b} = b^{bis}$; otherwise let $\underline{b} = b^{bis}$.

Iterate on *Step* Vb' until $\widetilde{b}_S = b_S^{bis}$ with a given degree of precision.

Hence, in our case, the completion of the bisection steps $Va' - Vb'$ does not guarantee finding the true solution because the convergence in the coefficient b_S was achieved for some sequence of the coefficients $(b_1, ..., b_{S-1})$ over which we had no control. Thus, at the end of the bisection step, we compute $\overset{\approx}{b}$ for next iteration by updating according to (14) and continue iterations until the convergence of all the coefficients, $\overset{\approx}{b} = b$. In some cases, our bisection and updating method can imply a substantial reduction in computational time. In Appendix A, we describe a numerical example where our bisection and updating method delivers a solution almost three times faster than the pure updating method. We have studied an application of our bisection and updating method only to a simple setting, in which the aggregate state space is summarized by one statistic and in which the ALM is given only by one equation. However, we expect this method to be also useful in more sophisticated settings including those in which decision rules are state contingent and in which the aggregate state space is represented by more than one statistic. Also, this method can be always used for fast solving a one-moment version of the model, so that the obtained solution is subsequently used as an initial guess for updating in several-moment versions of the model.

4 Calibration and algorithms parameters

We calibrate the model with aggregate uncertainty by using the following set of parameters.

Table 1: Benchmark calibration

Parameters	β	γ	α	δ	μ	$\bar{l}$	Δ^a
Values	0.99	1	0.36	0.025	0.15	1/0.9	0.01

The unemployment rate in the good and bad states is equal 4% and 10%, respectively. The transition probabilities are given in Table 2.

Table 2: Transition probabilities

$s, \varepsilon / s' \varepsilon'$	$1-\Delta^a, 0$	$1-\Delta^a, 1$	$1+\Delta^a, 0$	$1+\Delta^a, 1$
$1-\Delta^a, 0$	0.525	0.35	0.03125	0.09375
$1-\Delta^a, 1$	0.038889	0.836111	0.002083	0.122917
$1+\Delta^a, 0$	0.09375	0.03125	0.291667	0.583333
$1+\Delta^a, 1$	0.009115	0.115885	0.024306	0.850694

For the model with only idiosyncratic shocks, we assume that the productivity takes an average value $a_t = 1$ for all t, the employment takes values corresponding to the bad aggregate state $L^b = 0.9$; and the Markov chain for the individual shocks is given in Table 3.

Table 3: Transition probabilities. No aggregate shocks

$\varepsilon / \varepsilon'$	0	1
0	0.40	0.60
1	0.044445	0.955555

By taking into account the normalization $\bar{l} = 1/L^b = 1/0.9$, we have that production is given by $Y = K^\alpha$ and that the prices for capital and labor are, $r = \alpha K^{\alpha-1}$ and $w = (1-\alpha) K^{\alpha}$, respectively. In the case of exogenous interest rate, we assume that $K = 43$ and deduce the corresponding interest rate and wage.

We now discuss the choice of some algorithms parameters. First, the properties of the solution to the model will depend on the initial distribution of wealth assumed, and the effect of initial conditions does not vanish over time. Given this feature of the model, it is natural to take the initial distribution of wealth drawn from the ergodic distribution. However, such distribution is unknown before the model is solved. We therefore solve the model twice: we first assume that all agents have initial wealth equal to the steady state level of capital, and we then use the terminal distribution of wealth generated by the model as the initial distribution of wealth. We report the properties of the solution when the initial wealth is drawn from the ergodic distribution.

Second, to solve the individual problem, we are to choose the domain for assets $[k_{\min}, k_{\max}]$, the number of grid points and their placement in the domain and a specific interpolation procedure. By definition (3), the individual wealth is to be nonnegative, so we have $k_{\min} = 0$. Concerning $k_{\max}$, we are to choose a value which is sufficiently large in order not to be reached along simulations. We take $k_{\max} = 1000$, and in our subsequent simulations, the individual wealth never reached 500.

Finally, to evaluate the asset function outside the grid, we tried to use both linear and cubic (spline) polynomial interpolation. The cubic polynomial interpolation is two or three times slower but produces more accurate solution than the linear one. Given the restriction on computational cost, we therefore face a trade-off between a linear interpolation with a large number of points and a cubic interpolation with a small number of points. We run a number of experiments, and we concluded that the cubic interpolation with few points is superior to the linear one, especially, in areas where the decision rules are non-linear.

5 Results

In this section, we report the simulation results. We first study the individual problem with exogenous interest rate, we then consider the model with only idiosyncratic uncertainty, and we finally analyze the model with both idiosyncratic and aggregate uncertainty.

5.1 The individual problem

In order to solve the individual problem, we are first to decide how many grid points to take and where to place them in the domain. It is known since the work of Huggett (1993) and Aiyagari (1994) that the individual decision rules in this class of problems are non-linear near the liquidity constraints but are close to linear at large levels of capital. In order to obtain an accurate solution, one is to place many grid points at low levels of capital and few grid points at high levels of capital. We propose the following polynomial rule for placement of grid points

$$y(x) = \frac{x^m}{(0.5)^m} \text{ for } x = [0, 0.5], \tag{15}$$

where m is a degree of the polynomial. This function is normalized so that $y(0) = 0$ and $y(1) = 1$. If $m = 1$, the grid points are distributed uniformly, $y(x) = x$, and if $m > 1$, the points are more concentrates at low levels of capital than at high levels of capital. By increasing m, we can increase (decrease) concentration of grid points in the beginning (the end) of the interval. The placement of the grid points depending on the polynomial degree m is shown in Table 4 and is illustrated graphically in Figure 1 for the case of 20 grid points.

To see which degree m leads to the most accurate solution, we first compute an "accurate" solution by considering 100000 grid points uniformly placed in the interval $[k_{\min}, k_{\max}]$. We then compute the decision rules by considering 100 points placed according to (15) under different degrees m. In the upper part of Table 5, we report the average and the maximum percentage errors between the capital choice under the accurate solution and the 100 grid-point solutions. As we see, the smallest maximum error is achieved under the degree $m = 7$. In this table, we also report the level of wealth under which the maximum error is achieved. As we see, the largest errors are obtained under very low levels of wealth, 0.6-1.6 (i.e., in the area of non-linearity). We view the solution with 100-points as relatively accurate: the average error is about 0.0002% and the maximum error is about 0.09%. In Table 5, we also tried to see how the accuracy depends on the number of grid points, and we find that increasing the number of grid points from 100 to 400 increases the accuracy by about one order of magnitude.

5.2 Model with no aggregate uncertainty

We assess the accuracy of the solution in the model without aggregate uncertainty. To solve for an interest rate in such a model, we use a bisection procedure described in Section 3.1.1. To run simulations, we assume $N = 10000$ heterogeneous agents, we fix the length of time series to $T = 1100$, and we discard the first $T_1 = 100$ periods in order to eliminate the effect of initial conditions. We perform a bisection until 10^{-10} precision in the interest rate value was achieved. We again compute an "accurate" solution by taking a grid of 5000 points distributed according to a 7 degree polynomial, and we compare all other solutions to this accurate solution. In the upper part of Table 6, we report the average and the maximum errors and the difference between the steady state levels of capital under one fixed sequence of shocks. We see that a 100-point solution is still sufficiently accurate: the average error is about 0.005% and the maximum error is about 0.2%. The difference in the steady state levels of capital across the experiments is very small, namely, less than $1.5 \times 10^{-7}\%$.

We also study how the properties of the solution depend on a specific realization of shocks. We specifically compute the solution to the 100 point model by re-drawing shocks four times and compared the first solution to the other three solutions. The results are reported in the bottom part of Table 6. As we see, the obtained differences in the decision rules and the steady state capital stock across the four realizations are very small, $10^{-8}\%$, so we conclude that our choice ($N = 10000$ and $T = 1100$ with $T_1 = 100$ discarded periods) leads to solutions the properties of which do not significantly depend on specific realization of shocks.

5.3 Results

This section presents the results for the model with aggregate uncertainty. We solved the models by characterizing the aggregate state space with the mean of the wealth distribution, and also, with both the mean and the variance of the wealth distribution.

For the case of one moment (mean), the ALM are given by

$$\ln(K_{t+1}) = 0.1235 + 0.9657 \ln(K_t) \qquad R^2 = 0.999934$$

with the low aggregate shock, and by

$$\ln(K_{t+1}) = 0.1385 + 0.9631 \ln(K_t) \qquad R^2 = 0.999967$$

with the high aggregate shock. For the case of two moments (mean and variance), the ALM for aggregate (mean) capital are given by

$$\ln\left(K_{t+1}\right) = 0.1224 + 0.9661 \ln\left(K_t\right) \quad R^2 = 0.999935$$

with the low aggregate shock, and by

$$\ln\left(K_{t+1}\right) = 0.1398 + 0.9628 \ln\left(K_t\right) \quad R^2 = 0.999967$$

with the high aggregate shock; the ALM for the variance of capital, Σ, are given by

$$\ln\left(\Sigma_{t+1}\right) = 0.1745 - 0.0393 \ln\left(K_t\right) + 0.9958 \ln\left(\Sigma_t\right) \quad R^2 = 0.999699$$

with the low aggregate shock, and by

$$\ln\left(\Sigma_{t+1}\right) = 0.1705 - 0.0435 \ln\left(K_t\right) + 0.9958 \ln\left(\Sigma_t\right) \quad R^2 = 0.999920$$

with the high aggregate shock. Computational time required for solving the one-moment model was 1.31 hour, and computational time for solving the two-moment model was 5.23 hours.

To illustrate the solution to the individual problem, we present the following results. Figures 2 shows the decision rules for next-period capital and consumption as functions of the individual capital for the four possible combinations of values for aggregate and idiosyncratic shocks: Low-low, Low-high, High-low and High-high where the first and the second name corresponds to the aggregate and the individual shocks, respectively.

Furthermore, in Figure 3, we plot the ALM for two possible values of the aggregate shock; we draw a stationary wealth distribution obtained in the last period of the simulation; we present the time series of aggregate capital obtained for the ALM-NO case; and finally, we plot the error of the aggregate capital series which is the difference between the capital series in the ALM-NO and ALM-YES cases.

We subsequently carried out all the required tests, and we calculated all the statistics, as is indicated in the description of the problems. The results in the tables and the figures correspond to the case when the aggregate state space is characterized only with the mean. In the ALM-NO case, we simulate the time series for a panel of $N = 10000$ agents using the individual decision rules, and we average the individual quantities to compute the aggregate quantities. In the ALM-YES case, we use the ALM to compute the time series of aggregate capital, and we use the individual decision rules to compute the time series for one agent.

The results about risk sharing for both ALM-NO and ALM-YES cases are provided in Tables 7. Prices (interest rate and wage rate) are reported in Table 8. As far as the business cycle statistics are concerned, the standard deviations of and the cross-correlation at leads and lags between aggregate variables are provided in Tables 9, 10, 11, and 12. In Table 13, we present the results of the accuracy test about the Euler equation errors on a simulated time path for the ALM-NO case (for the first 10 agents of the sample) and for the ALM-YES case (for one agent). Also, in this table, we illustrate the accuracy of aggregate policy rule by providing the maximum and the average absolute errors of the difference between the time series of aggregate capital for the ALM-NO and ALM-YES cases. The Denhaan-Marcet statistic regards is reported in Table 14: we show the results for the ALM-NO case (the test was replicated 100 times using the numerical solution for the first 10 agents of the sample), and we also show the results for the ALM-YES case (with only one replication). Finally, the statistics of the wealth distribution obtained for the ALM-NO case are given in Table 15.

6 Conclusion

In this paper, we describe an algorithm for solving Krusell and Smith's (1998) model with a continuum of agents where savings are made through capital. To solve the individual problem, we use an Euler equation algorithm iterating on a grid of prespecified points. To solve for the ALM, we summarize the economy's state by the first two moments of the wealth distribution and use Monte Carlo simulation. Our algorithm is relatively fast, especially, if the ALM is computed with our bisection and updating method. The solutions delivered by our algorithm are sufficiently accurate, as the accuracy checks show. Overall, our results are close to those in Young (2003), which is not surprising given that we characterize the aggregate state space with the same set of statistics as he does. It would be of interest to see the performance of our method in the context of Model C, where, as found by Young (2003) and Reiter (2003), no single aggregate statistics is sufficient for accurate forecasting of the future prices.

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7 Appendix A

In this appendix, we describe a numerical example where we compare the performance of Krusell's and Smith bisection method described in Section 3.3.1 with an updating and bisection method described in Section 3.3.2. To carry out the comparison, we consider Krusell and Smith's (1998). We represent the aggregate economy state by the mean of the wealth distribution and parameterize ALM (13) as follows:

$$\ln(K_{t+1}) = b_1 + b_2 a_t + b_3 \ln(K_t). \tag{16}$$

This specification is more restrictive than the one considered in Krusell and Smith (1998) because it forces the coefficient on $\ln(K_t)$ to be the same with both high and low aggregate productivity shocks, however, as we will see, it still leads to an accurate approximation.

We consider the model with $\gamma = 1$ and solve it first by using Krusell and Smith's (1998) updating described in Section 3.3.1. and then by using the bisection and updating method that we propose in Section 3.3.2 for an identical set of the parameters including the realizations of the individual and aggregate shocks. In both cases, we use the same updating parameter of $\eta_b = 0.5$ and target the same degree of precision, namely, that $\overset{\approx}{b}$ and b differ by not more than 10^{-5} according to the least-square norm. We set the precision of bisection to a relatively low value such that $\widetilde{b}_S$ and b_S^{bis} differ by less than 10^{-3} (since bisection has to be implemented after each updating, it is not essential to have a high degree of precision on each single step but only to achieve high precision at the end of the algorithm). Our initial guess for the coefficients was $b = (0, 0, 1)$. We summarize the results of this experiment below:

Updating	Bisection and updating
$b_1 = 0.08799863$	$b_1 = 0.08800197$
$b_2 = 0.00532372$	$b_2 = 0.00532067$
$b_3 = 0.96321271$	$b_3 = 0.96321262$
$R^2 = 0.99994110$	$R^2 = 0.99994110$
Computational time = 34.24 min	Computational time = 13.44 min

There are two noteworthy results here. First, our parameterization (16) delivers a solution which is comparable in accuracy to the one obtained under Krusell and Smith's (1998) parameterization of two state-contingent ALM. (Under their parameterization, we have two state contingent values of $R^2 = 0.99994989$ and $R^2 = 0.99992851$ with the high and the low aggregate productivity shocks, respectively). Second, under the identical parameterization (16) and given the same degree of accuracy, our bisection and updating method delivers a solution almost three times faster than the pure updating. For the sake of comparison, we shall also mention that it took us more than an hour to solve this model by using pure updating under the state-contingent ALM, as in Krusell and Smith (1998).

TABLE 4
Grid-point placement depending on polynomial degree

x	x	x^3	x^5	x^7	x^9
0	0	0	0	0	0
0,052632	0,052632	0,00014579	4,0386e-007	1,1187e-009	3,099e-012
0,10526	0,10526	0,0011664	1,2924e-005	1,432e-007	1,5867e-009
0,15789	0,15789	0,0039364	9,8138e-005	2,4467e-006	6,0997e-008
0,21053	0,21053	0,0093308	0,00041355	1,8329e-005	8,1238e-007
0,26316	0,26316	0,018224	0,0012621	8,7401e-005	6,0527e-006
0,31579	0,31579	0,031491	0,0031404	0,00031317	3,123e-005
0,36842	0,36842	0,050007	0,0067877	0,00092132	0,00012505
0,42105	0,42105	0,074646	0,013234	0,0023461	0,00041594
0,47368	0,47368	0,10628	0,023848	0,0053508	0,0012006
0,52632	0,52632	0,14579	0,040386	0,011187	0,003099
0,57895	0,57895	0,19405	0,065042	0,021801	0,0073072
0,63158	0,63158	0,25193	0,10049	0,040086	0,01599
0,68421	0,68421	0,32031	0,14995	0,070199	0,032863
0,73684	0,73684	0,40006	0,21721	0,11793	0,064028
0,78947	0,78947	0,49205	0,30668	0,19115	0,11913
0,84211	0,84211	0,59717	0,42348	0,30031	0,21296
0,89474	0,89474	0,71629	0,57342	0,45906	0,3675
0,94737	0,94737	0,85027	0,76312	0,68491	0,61471
1	1	1	1	1	1

TABLE 5
Decision rule errors depending on polynomial degree and number of grid points

	Average error compared to the accurate solution	Maximum error compared to the accurate solution	Wealth under maximum error
100 grid point			
Degree of polynomial			
1	0,0023444	0,34711	1.6
2	6,382e-006	0,010158	0.6
3	2,8005e-006	0,0037153	1.6
4	1,7269e-006	0,0019896	1.6
5	1,6978e-006	0,0018806	0.6
6	1,7102e-006	0,0016388	1.6
7	1,9658e-006	0,00086033	0.6
8	2,2776e-006	0,0012321	0.6
9	2,5752e-006	0,0011811	0.9
Polynomial of degree 7			
Number of the grid points			
100	1,9658e-006	0,00086033	0.6
200	5,5139e-007	0,00078441	1.0
300	2,0893e-007	0,00021801	1.0
400	1,0151e-007	0,00015021	2.5

TABLE 6
Decision rule errors depending on polynomial degree and number of grid points

	Average error compared to the accurate solution	Maximum error compared to the accurate solution	Percentage difference in capital stocks

100 grid point

Degree of polynomial			
1	1,318e-011	3,4227e-011	1,0926e-011
2	9,7232e-011	1,9258e-010	5,0491e-011
3	2,0634e-010	4,0765e-010	1,0662e-010

Polynomial of degree 7

Number of the grid points			
100	4,2835e-005	0,0024234	1,4717e-009
200	1,6516e-005	0,00090135	1,4389e-009
300	1,039e-005	0,00044333	1,4359e-009
400	7,1918e-005	0,00039331	1,4717e-009

TABLE 7
Risk sharing statistics

	ALM-NO		ALM-YES
	Average across agents	Standard deviation	Statistic
Correlation of individual and aggregate consumption	0,4215	0,15806	0,44747
Correlation of individual consumption and aggregate income	0,28869	0,095484	0,30769
Correlation of individual consumption and aggregate wealth	0,4064	0,15831	0,42966
Correlation of individual consumption and individual income	0,54468	0,072166	0,57525
Correlation of individual consumption and individual wealth	0,91109	0,041365	0,93777
Standard deviation of individual consumption	0,11043		0,11478
Standard deviation of individual capital	7,3558		8,1248
Autocorrelation of individual consumption	0,98152	0,013052	0,98864
1 lag	0,96333	0,024117	0,97549
2 lags	0,94572	0,03352	0,96098
3 lags	0,99647	0,0019323	0,99731
Autocorrelation of individual capital	0,98979	0,0055511	0,99188
1 lag	0,98166	0,0099143	0,98489
2 lags	0,4215	0,15806	0,44747
3 lags	0,28869	0,095484	0,30769

TABLE 8
Prices

	ALM-NO		ALM-YES	
	Average	Standard deviation	Average	Standard deviation
Interest rate	0,034894	0,0010552	0,034896	0,0010561
Wage rate	2,384	0,018091	2,3839	0,018055
Autocorrelation of interest rate				
1 lag	0,7364		0,7381	
2 lags	0,51821		0,52035	
3 lags	0,3556		0,3564	
Autocorrelation of wage rate				
1 lag	0,97531		0,97527	
2 lags	0,9439		0,94363	
3 lags	0,9095		0,90893	

TABLE 9
Business cycle statistics

	ALM-NO	ALM-YES
Average income	0,11889	0,12716
Average consumption	0,044095	0,042835
Average investment	0,10494	0,1049

TABLE 10
Cross correlation of income and consumption at leads and lags

ALM-NO

	y_t	y_{t-1}	y_{t-2}	y_{t-3}	c_t	c_{t-1}	c_{t-2}	c_{t-3}
y_t	1	0,8019	0,63418	0,50755	0,68522	0,558	0,45468	0,37479
y_{t-1}		1	0,80161	0,63241	0,71034	0,68421	0,55681	0,45307
y_{t-2}			1	0,79998	0,71611	0,70939	0,68316	0,55527
y_{t-3}				1	0,71139	0,71581	0,70923	0,68331
c_t					1	0,98237	0,95607	0,92463
c_{t-1}						1	0,98236	0,95592
c_{t-2}							1	0,98227
c_{t-3}								1

ALM-YES

	y_t	y_{t-1}	y_{t-2}	y_{t-3}	c_t	c_{t-1}	c_{t-2}	c_{t-3}
y_t	1	0,79687	0,62504	0,49286	0,6434	0,51944	0,41972	0,34269
y_{t-1}		1	0,79663	0,6233	0,67902	0,64235	0,51823	0,41816
y_{t-2}			1	0,79504	0,69336	0,67804	0,64126	0,51675
y_{t-3}				1	0,69515	0,69319	0,678	0,64151
c_t					1	0,98567	0,96212	0,93277
c_{t-1}						1	0,98566	0,96201
c_{t-2}							1	0,9856
c_{t-3}								1

TABLE 11
Cross correlation of income and investment at leads and lags

ALM-NO

	y_t	y_{t-1}	y_{t-2}	y_{t-3}	i_t	i_{t-1}	i_{t-2}	i_{t-3}
y_t	1	0,80156	0,6336	0,50684	0,943	0,75164	0,58751	0,46475
y_{t-1}		1	0,80129	0,63188	0,68231	0,94286	0,75133	0,5861
y_{t-2}			1	0,79971	0,4689	0,68194	0,94271	0,75007
y_{t-3}				1	0,31172	0,4665	0,67951	0,9427
i_t					1	0,74176	0,52724	0,36911
i_{t-1}						1	0,74165	0,52525
i_{t-2}							1	0,73967
i_{t-3}								1

ALM-YES

	y_t	y_{t-1}	y_{t-2}	y_{t-3}	i_t	i_{t-1}	i_{t-2}	i_{t-3}
y_t	1	0,79687	0,62504	0,49286	0,94989	0,75376	0,58586	0,4572
y_{t-1}		1	0,79663	0,6233	0,68903	0,94979	0,75352	0,58443
y_{t-2}			1	0,79504	0,47481	0,68873	0,94966	0,75225
y_{t-3}				1	0,3138	0,47246	0,68635	0,94964
i_t					1	0,74228	0,52782	0,36605
i_{t-1}						1	0,74223	0,52588
i_{t-2}							1	0,74029
i_{t-3}								1

TABLE 12
Cross correlation of consumption and investment at leads and lags

ALM-NO

	c_t	c_{t-1}	c_{t-2}	c_{t-3}	i_t	i_{t-1}	i_{t-2}	i_{t-3}
c_t	1	0,74176	0,52724	0,36911	0,943	0,68231	0,4689	0,31172
c_{t-1}		1	0,74165	0,52525	0,75164	0,94286	0,68194	0,4665
c_{t-2}			1	0,73967	0,58751	0,75133	0,94271	0,67951
c_{t-3}				1	0,46475	0,5861	0,75007	0,9427
i_t					1	0,80156	0,6336	0,50684
i_{t-1}						1	0,80129	0,63188
i_{t-2}							1	0,79971
i_{t-3}								1

ALM-YES

	c_t	c_{t-1}	c_{t-2}	c_{t-3}	i_t	i_{t-1}	i_{t-2}	i_{t-3}
c_t	1	0,98567	0,96212	0,93277	0,37187	0,42059	0,44718	0,46129
c_{t-1}		1	0,98566	0,96201	0,22741	0,37029	0,41901	0,44697
c_{t-2}			1	0,9856	0,1161	0,22571	0,36859	0,41894
c_{t-3}				1	0,03468	0,11408	0,22363	0,36884
i_t					1	0,74228	0,52782	0,36605
i_{t-1}						1	0,74223	0,52588
i_{t-2}							1	0,74029
i_{t-3}								1

TABLE 13
Absolute percentage error

	Average error	**Maximum error**
Euler equation errors on a simulated time path (ALM-NO)	0,0021769	0,021991
Euler equation errors on a simulated time path (ALM-YES)	0,0086893	0,10129
Aggregate policy rule errors	0,00060847	0,0021909

TABLE 14
Den Haan and Marcet statistics

ALM-NO		**ALM-YES**	
Fraction of times below 2.5% critical value	0	Statistic for one agent	5,8149
Fraction of times above 97.5% critical value	43	Low critical value	1,2373
		High critical value	14,449

TABLE 15
Wealth distribution (ALM-NO)

	Unconditional	**Conditional on low shock**	**Conditional on high shock**
Fraction of times agent is at constraint	3,2062e-005	5,1954e-005	2,2478e-005
5th percentile of wealth distribution	13,04	12,847	13,188
10th percentile of wealth distribution	16,201	16,017	16,342
Average capital stock	39,934	39,715	40,103
Moment 2	0,86301	0,86523	0,86131
Moment 3	1,2616	1,2645	1,2595
Moment 4	1,7022	1,7061	1,6991
Moment 5	2,0751	2,0798	2,0714

Figure 1. Power grid functions.

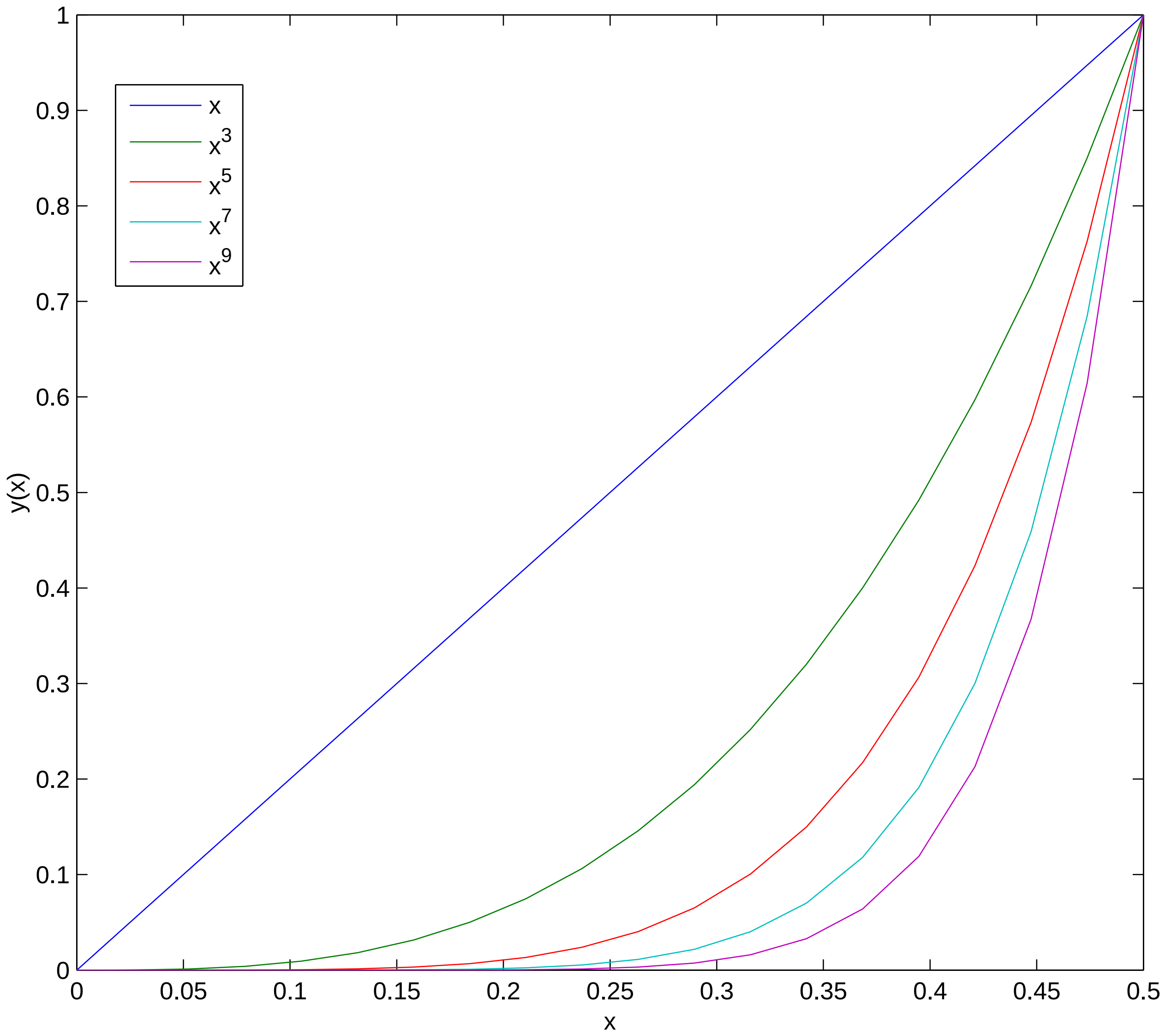

Figure 2. Individual decision rules.

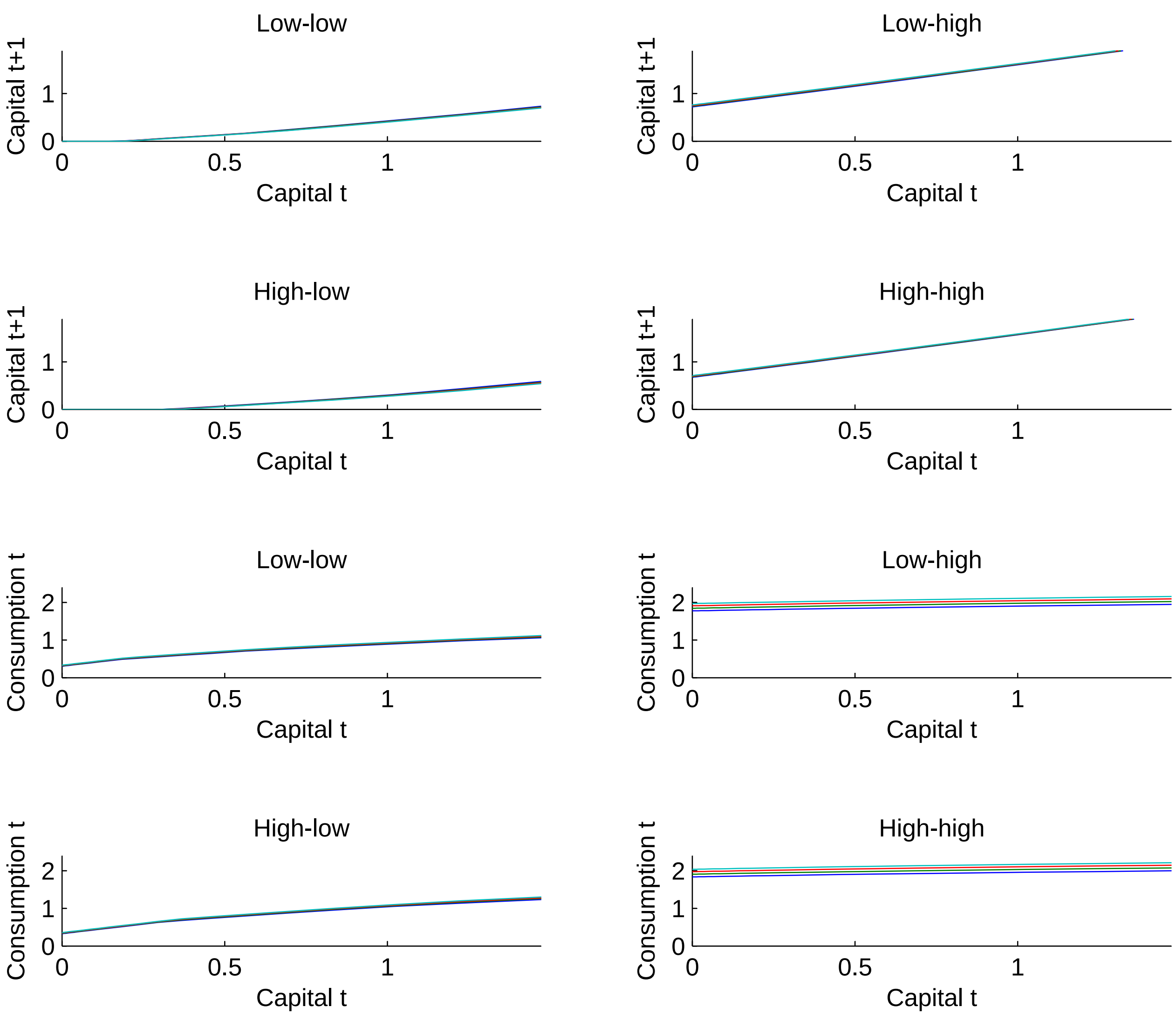

Figure 3. Properties of the solution.