Comparing numerical solutions of models with heterogeneous agents (Model A): a simulation-based PEA

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Model A

- Simulation-based Parameterized Expectation Algorithm (PEA) by *den Haan and Marcet (JBES, 1990).*
- Modifications to the PEA.
  - to enhance convergence: *Maliar and Maliar (JBES, 2003);*
  - to reduce the cost of intratemporal FOCs: *Maliar and Maliar (Computational Economics, 2005).*
- We study only models with heterogeneous fundamentals A5, A7, A8.
- We, intentionally, do not use simplifying analytical results (e.g., aggregation).

In the presentation, to illustrate the method, we

- focus on Model A5 (no labor);
- study only two countries ($N = 2$);
- neglect the adjustment costs;

$$\max_{\left\{c_t^1, c_t^2, k_{t+1}^1, k_{t+1}^1\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\sum_{n=1}^{2} \tau^n \frac{\left(c_t^n\right)^{1-1/\gamma_n} - 1}{1 - 1/\gamma_n} \right]$$

s.t.

$$\sum_{n=1}^{2} c_t^n + \sum_{n=1}^{2} k_{t+1}^n = (1 - \delta) \sum_{n=1}^{2} k_t^n + \sum_{n=1}^{2} a_t^n A f\left(k_t^n\right).$$

- This model contains all relevant features to discuss.

- Recall the PEA in a similar one-country model:

$$c_t^{-1/\gamma} = \beta E_t \left\{ c_{t+1}^{-1/\gamma} \left[1 - \delta + a_{t+1} A f_1 \left(k_{t+1} \right) \right] \right\},$$
$$c_t + k_{t+1} = (1-\delta) k_t + a_t A f(k_t).$$

- Parameterize the expectation by

$$E_t \left\{ (c_{t+1})^{1-1/\gamma} \left[1 - \delta + a_{t+1} A f_1 \left(k_{t+1} \right) \right] \right\}$$
$$\equiv \Phi(k_t, a_t, v)$$

where v is a vector of parameters.

- Substituting $(c_t)^{-1/\gamma} = \beta \Phi(k_t, a_t, v)$ into budget constraint yields:

$$k_{t+1} = (1-\delta) k_t + a_t A f(k_t) - \left[\beta \Phi(k_t, a_t, v) \right]^{-\gamma}.$$

The algorithm is as follows:

1. Draw shocks $\{a_t\}_{t=1}^T$.
2. Fix some v and simulate a sequence $\{k_{t+1}\}_{t=1}^T$ from

   $$k_{t+1} = (1-\delta)\, k_t + a_t A f(k_t) - [\beta \Phi(k_t, a_t, v)]^{-\gamma}.$$

3. Restore consumption $c_t = [\beta \Phi(k_t, a_t, v)]^{-\gamma}$.
4. Check if $\{c_t, k_{t+1}\}_{t=1}^T$ satisfies the Euler equation,

   $$E_t \left\{ (c_{t+1})^{1-1/\gamma} [1 - d + a_{t+1} A f_1(k_{t+1})] \right\} \equiv \Phi(k_t, a_t, v),$$

   by running a NLLS regression.

Iterate on v until convergence.

Two country model

$$\left(c_t^1\right)^{-1/\gamma_1} = \beta E_t \left\{ \left(c_{t+1}^1\right)^{-1/\gamma_1} \left[1 - \delta + a_{t+1}^1 A f_1 \left(k_{t+1}^1\right)\right] \right\},$$

$$\left(c_t^2\right)^{-1/\gamma_2} = \beta E_t \left\{ \left(c_{t+1}^2\right)^{-1/\gamma_2} \left[1 - \delta + a_{t+1}^2 A f_1 \left(k_{t+1}^2\right)\right] \right\},$$

$$\tau^1 \left(c_t^1\right)^{-1/\gamma_1} = \tau^2 \left(c_t^2\right)^{-1/\gamma_2}$$

$$\sum_{n=1}^{2} c_t^n + \sum_{n=1}^{2} k_{t+1}^n = (1-\delta) \sum_{n=1}^{2} k_t^n + \sum_{n=1}^{2} a_t^n A f\left(k_t^n\right)$$

Observation 1. We cannot parameterize expectations as before:

- three equations to identify two consumption, $\left(c_t^1\right)^{-1/\gamma_1} = \beta\Phi_1(\cdot, v_1)$, $\left(c_t^2\right)^{-1/\gamma_2} = \beta\Phi_2(\cdot, v_2)$, and $\tau^1 \left(c_t^1\right)^{-1/\gamma_1} = \tau^2 \left(c_t^2\right)^{-1/\gamma_2}$;
- and one budget constraint to identify two capitals, k_{t+1}^1, k_{t+1}^2.

To deal with this problem, we rewrite the Euler equations as

$$k_{t+1}^{1} = \frac{\beta E_t \left\{ \left(c_{t+1}^{1}\right)^{-1/\gamma_1} \left[1 - d + a_{t+1}^{1} A f_1 \left(k_{t+1}^{1}\right)\right] \right\} k_{t+1}^{1}}{\left(c_t^{1}\right)^{-1/\gamma_1}} = \beta \Phi_1 \left(\cdot, v_1\right)$$

$$k_{t+1}^{2} = \frac{\beta E_t \left\{ \left(c_{t+1}^{2}\right)^{-1/\gamma_2} \left[1 - d + a_{t+1}^{2} A f_1 \left(k_{t+1}^{2}\right)\right] \right\} k_{t+1}^{2}}{\left(c_t^{2}\right)^{-1/\gamma_2}} = \beta \Phi_2 \left(\cdot, v_2\right)$$

- Similar approach is used in Marcet and Lorenzoni (1999, eds. R. Marimon and A. Scott).
- Maliar and Maliar (Computational Economics, 2005) show that specific parameterization can affect much the computation cost.

To parameterize the capital stock, we use an exponentiated quadratic polynomial

$$\begin{bmatrix} k_{t+1}^1 \\ k_{t+1}^2 \end{bmatrix} = \exp\left\{ \begin{bmatrix} v_0^1 \\ v_0^2 \end{bmatrix} + \begin{bmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{bmatrix} \begin{bmatrix} \log\left(k_t^1\right) \\ \log\left(k_t^2\right) \end{bmatrix} \right.$$

$$+ \begin{bmatrix} v_3^1 & v_3^1 \\ v_3^2 & v_4^2 \end{bmatrix} \begin{bmatrix} \log\left(a_t^1\right) \\ \log\left(a_t^2\right) \end{bmatrix} + \begin{bmatrix} v_4^1 & v_4^1 \\ v_4^2 & v_4^2 \end{bmatrix} \begin{bmatrix} \log^2\left(k_t^1\right) \\ \log^2\left(k_t^2\right) \end{bmatrix}$$

$$\left. + \begin{bmatrix} v_5^1 & v_5^1 \\ v_5^2 & v_5^2 \end{bmatrix} \begin{bmatrix} \log^2\left(a_t^1\right) \\ \log^2\left(a_t^2\right) \end{bmatrix} \right\},$$

- If we fix all v's, we can simulate the capital series $\left\{k_{t+1}^1, k_{t+1}^2\right\}_{t=1}^{T}$ directly from the above equation.
- We do not include cross terms.

Observation 2. We do not have a closed-form expression for intratemporal choice:

$$\tau^1 \left(c_t^1\right)^{-1/\gamma_1} = \tau^2 \left(c_t^2\right)^{-1/\gamma_2},$$

$$\sum_{n=1}^{2} c_t^n = (1-\delta)\sum_{n=1}^{2} k_t^n + \sum_{n=1}^{2} a_t^n A f\left(k_t^n\right) - \sum_{n=1}^{2} k_{t+1}^n$$

$$\Longrightarrow c_t^1 + \left[\frac{\tau_1}{\tau_2}\left(c_t^1\right)^{-1/\gamma_1}\right]^{-\gamma_2} = (1-\delta)\sum_{n=1}^{2} k_t^n + ...$$

- If we compute c_t^1, c_t^2 by a numerical solver at each date, it can be costly. *For example, if one simulation is* $T = 10000$*, and we make* 1000 *iterations, the solver is used* 10^7 *times.*

- The issue of intratemporal choice is particularly important for simulation-based methods.

Two alternatives for the intratemporal choice

1. Compute the relation between the individual and aggregate consumption outside of the iterative cycle *(Maliar and Maliar, 2005, Computational Economics)*.

   - Take a grid for values for aggregate consumption $C_t = c_t^1 + c_t^2$ such that $C_m \in \{C_1, C_2, ..., C_M\}$.
   - Define the grid function $c^1(C_m)$, $m = 1, ..., M$ by finding c^1 that solves

   $$c_t^1 + \left[\frac{\tau^1}{\tau^2}\left(c_t^1\right)^{-1/\gamma_1}\right]^{-\gamma_2} = C_m.$$

   - Within the iterative cycle, compute c_t^1, c_t^2 at each date t by interpolation.

2. Restore the intratemporal choice by iterating on consumption allocations.

- Write the optimality conditions as

$$\widetilde{c}_t^1 = \sum_{n=1}^{2} \left[(1-\delta) k_t^n + a_t^n A (k_t^n)^\alpha - k_{t+1}^n \right] - c_t^2,$$

$$\widetilde{c}_t^2 = \left[\frac{\tau^1}{\tau^2} \left(c_t^1 \right)^{-1/\gamma_1} \right]^{-\gamma_2}.$$

- For given $\left\{k_{t+1}^1, k_{t+1}^2\right\}_{t=1}^T$, take some series $\left\{c_t^1, c_t^2\right\}_{t=1}^T$ (e.g., steady state), and compute $\left\{\widetilde{c}_t^1, \widetilde{c}_t^2\right\}_{t=1}^T$.
- Iterate on the consumption series until convergence.

Convergence issue

PEA is not a contraction mapping method and thus, does not guarantee finding a solution.

- *If approximation is inaccurate, the simulated series are non-stationary;*
- *The regression does not work appropriately;*
- *The PEA breaks down.*

Maliar and Maliar (JBES, 2003) propose to enhance the convergence by restricting k^n_{t+1} to be within bounds

$$\begin{aligned} \underline{k}(i) &= k_{ss}\exp(-\lambda i), \\ \overline{k}(i) &= k_{ss}(2-\exp(-\lambda i)), \end{aligned}$$

where i is iteration, and $\lambda > 0$ is a parameter.

- On first iteration $(i = 0)$, we have $k^n_t(0) = k_{ss}$ for all t.
- On subsequent iterations, $\underline{k}(i)$ and $\overline{k}(i)$ gradually move approaching 0 and $2k_{ss}$, respectively.

The bounds induce stationarity on initial iterations and become irrelevant when the solution is refined.

Parameterization

Table 1. Common parameters.

α	β	δ	ρ	σ	ϕ	ξ
0.36	0.99	0.025	0.95	0.01	10	2

Table 2. Country-specific parameters.

A5	A7	A8
$(\gamma_m, \gamma_M) = (0.25, 1)$	$(\gamma_m, \gamma_M) = (0.25, 0.5)$	$(\gamma_m, \gamma_M) = (0.2, 0.4)$ $(\chi_m, \chi_M) = (0.75, 0.9)$ $(\mu_m, \mu_M) = (-0.3, 0.3)$

- $N = 2, 6, 10$.
- Two lengths of simulations, $T = 3000$ and $T = 10000$.
- Matlab; Pentium 4 PC with 3.33GHz processor.

Figure 1. Model A8 with 2 countries: time series solution.

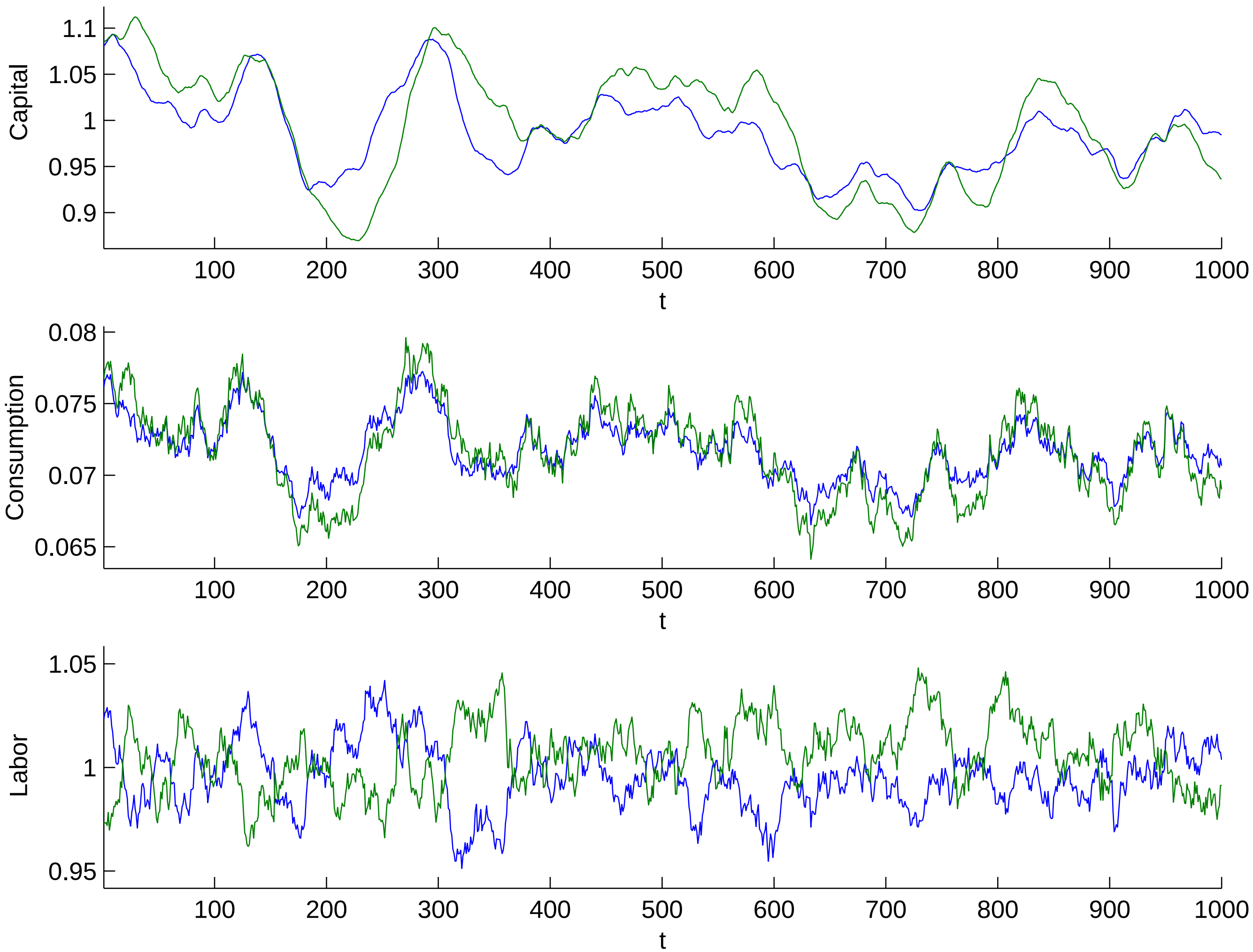


Figure 2. Model A8 with 6 countries: time series solution.

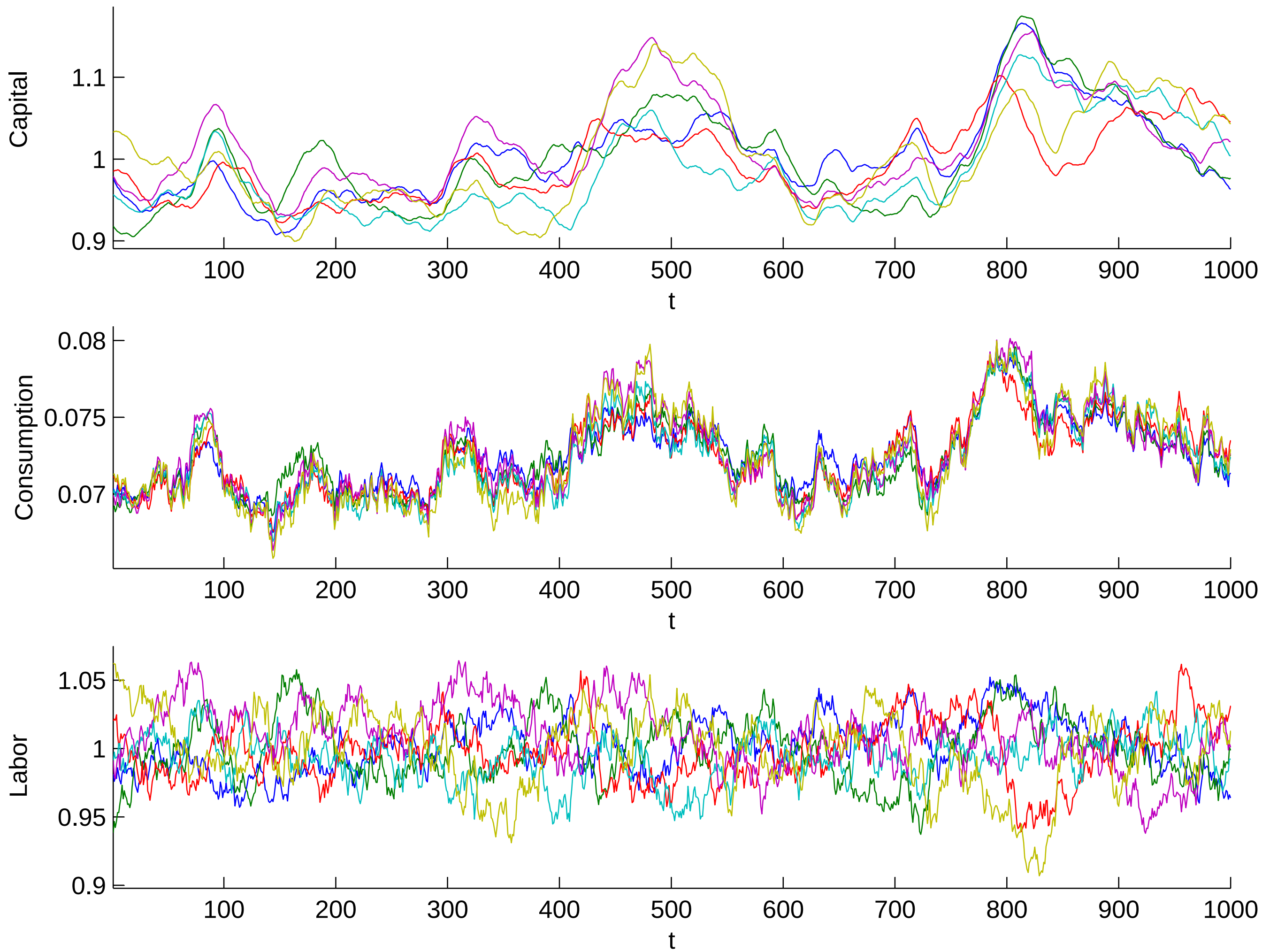
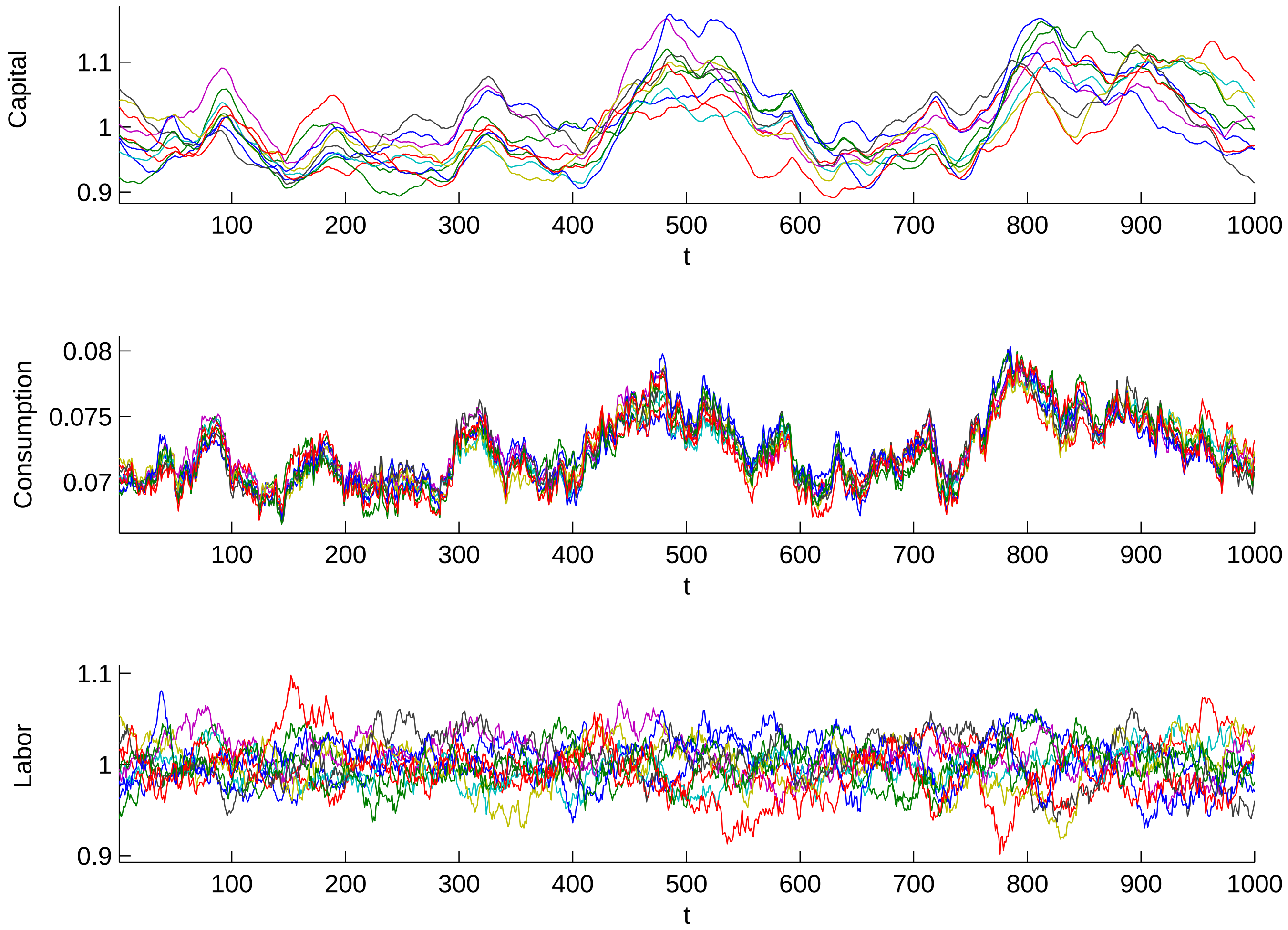


Figure 3. Model A8 with 10 countries: time series solution.



The accuracy measures

- Under our method, the budget constraint is satisfied exactly;
- The intratemporal choice is approximated very accurately, 10^{-10}.
- *Thus, we focus only on the Euler equation error*

$$\epsilon_t^n = \frac{\beta E_t \left\{ u_1^n \left(c_{t+1}^n, l_{t+1}^n \right) \left[\theta_{t+1}^n + a_{t+1}^n A f_1^n \left(k_{t+1}^n, l_{t+1}^n \right) \right] \right\}}{u_1^n \left(c_t^n, l_t^n \right) \omega_t^n} - 1.$$

- Removed first 100 observations.
- $N = 2$, we use the product four-point Gauss-Hermite integration, and for $N = 6$ and $N = 10$, we use a monomial formula of degree 3.

Table 3. The results for Model A5.

	$N = 2$		$N = 6$		$N = 10$	
Sim. length	3000	10000	3000	10000	3000	10000
T (solution)	1m15s	4m24s	27m09s	1h03m	8h15m	14h33m
T(test 1)	0.63s	0.63s	1.76s	1.86s	42.20s	42.78s
$\epsilon^{\max}$ (r_1)	3(-4)	2(-4)	4(-4)	4(-4)	8(-4)	5(-4)
$\epsilon^{\max}$ (r_2)	4(-3)	4(-4)	5(-3)	2(-3)	5(-3)	2(-3)
$\epsilon^{\max}$ (r_3)	2(-2)	5(-3)	3(-2)	9(-3)	3(-2)	1(-2)
T(test 2)	0.23s	0.24s	0.56s	0.63s	14.05s	14.08s
$\overline{\epsilon}$	2(-4)	2(-4)	3(-4)	2(-4)	4(-4)	3(-4)
$\epsilon^{\max}$	1(-3)	5(-4)	2(-3)	1(-3)	2(-3)	2(-3)
T(test 3)	0.16s	0.55s	8.25s	28.95s	1m17s	4m26s
$m^{\min}$	4.75	5.91	71.74	54.21	268.53	182.24
$m^{\max}$	7.91	16.60	89.52	79.95	302.12	236.94

Table 4. The results for Model A7.

	$N=2$		$N=6$		$N=10$	
Sim. length	3000	10000	3000	10000	3000	10000
T (solution)	4m15s	8m56s	44m12s	1h10m	9h12m	13h48m
T(test 1)	15.94s	16.16s	5m32s	5m15s	3h25m	4h05m
$\epsilon^{\max}$ (r_1)	7(-4)	2(-4)	1(-3)	4(-4)	1(-3)	7(-4)
$\epsilon^{\max}$ (r_2)	7(-3)	5(-4)	1(-2)	2(-3)	5(-3)	3(-3)
$\epsilon^{\max}$ (r_3)	5(-2)	5(-3)	8(-2)	1(-2)	3(-2)	2(-2)
T(test 2)	5.39s	5.42s	1m53s	1m46s	1h19m	1h40m
$\overline{\epsilon}$	2(-4)	1(-4)	4(-4)	3(-4)	5(-4)	3(-4)
$\epsilon^{\max}$	1(-3)	5(-4)	2(-3)	1(-3)	3(-3)	2(-3)
T(test 3)	0.13s	0.59s	8.78s	28.99s	1m20s	5m15s
$m^{\min}$	9.97	23.45	73.27	98.74	200.69	211.63
$m^{\max}$	10.71	27.13	98.85	131.16	252.19	266.12

Table 5. The results for Model A8.

	$N=2$		$N=6$		$N=10$	
Sim. length	3000	10000	3000	10000	3000	10000
T (solution)	4.07m	12.03m	24m18s	1h24m	3h21m	4h34m
T(test 1)	30.86s	30.45s	1h21m	2h01m	5h26m	24h17m
$\epsilon^{\max}$ (r_1)	4(-4)	3(-4)	7(-4)	7(-4)	8(-4)	8(-4)
$\epsilon^{\max}$ (r_2)	3(-3)	6(-4)	2(-3)	2(-3)	4(-3)	2(-3)
$\epsilon^{\max}$ (r_3)	2(-2)	7(-3)	2(-2)	1(-2)	4(-2)	1(-2)
T(test 2)	10.36s	10.23s	42m05s	42m41s	1h56m	8h18m
$\overline{\epsilon}$	3(-4)	2(-4)	6(-4)	4(-4)	7(-4)	4(-4)
$\epsilon^{\max}$	1(-3)	7(-4)	3(-3)	2(-3)	4(-3)	2(-3)
T(test 3)	0.13s	0.63s	8.38s	25.45s	1m05s	4m28s
$m^{\min}$	16.23	50.88	88.07	104.38	215.10	222.39
$m^{\max}$	16.44	57.21	112.19	131.45	245.69	262.82

The main regularities:

- The cost increases around 2 times from $T_1 = 3000$ to $T_2 = 10000$ and errors decrease up to several times.
- In the ergodic set, the errors are $10(-4)$, but, the accuracy is reduced in the tails $10(-2)$.
- For a fixed length of simulation, the accuracy is reduced as the number of countries increases: the same number of observations to identify a larger number of parameters.
- The Denhaan-Marcet statistics are generally above the critical interval.

Conclusion:

- We solve Models A5, A7, A8 by using a simple simulation-based PEA.
- The algorithm takes few minutes to solve a two-country model, about half an hour to solve a six-country model and about 6 hours to solve a ten-country model. (In fact, we can increase the speed).
- The solutions are sufficiently accurate: in the ergodic distribution, the maximum errors do not exceed 10^{-3}, i.e., 0.1%.
- This magnitude of errors is not essential for many economically relevant applications such as second-moment properties.