Computing first and second order approximations of DSGE models with DYNARE

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CEPREMAP

DSGE models

$$E_t \left\{ f(y_{t+1}, y_t, y_{t-1}, u_t) \right\} = 0$$

$$\begin{aligned} u_t &= \sigma \epsilon_t \\ E(\epsilon_t) &= 0 \\ E(\epsilon_t \epsilon_t') &= \Sigma_\epsilon \end{aligned}$$

- y : vector of endogenous variables
- u : vector of exogenous stochastic shocks
- σ : stochastic scale variable
- ϵ : auxiliary random variables

Remarks

- The exogenous shocks may appear only at the current period
- There is no deterministic exogenous variables
- Not all variables are necessarily present with a lead and a lag
- Generalization to leads and lags on more than one period

Solution function

$$y_t = g(y_{t-1}, u_t, \sigma)$$

Then,

$$\begin{aligned} y_{t+1} &= g(y_t, u_{t+1}, \sigma) \\ & g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma) \\ F(y_{t-1}, u_t, u_{t+1}, \sigma) &= f(g(g(y_{t-1}, u_t, \sigma), u_{t+1}, \sigma), g(y_{t-1}, u_t, \sigma), y_{t-1}, u_t) \end{aligned}$$

$$E_t \left\{ F(y_{t-1}, u_t, u_{t+1}, \sigma) \right\} = 0$$

Steady state

A deterministic steady state, $\bar{y}$, for the model satisfies

$$f(\bar{y}, \bar{y}, \bar{y}, 0) = 0$$

A model can have several steady states, but only one of them will be used for approximation.
Furthermore,

$$\bar{y} = g(\bar{y}, 0, 0)$$

First order approximation

Around $\bar{y}$:

$$E_t\left\{F^{(1)}(y_{t-1},u_t,u_{t+1},\sigma)\right\} =$$

$$E_t\Big\{f(\bar{y},\bar{y},\bar{y},0) + f_{y_+}\left(g_y\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+g_u u'+g_\sigma\sigma\right)$$

$$+f_{y_0}\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+f_{y_-}\hat{y}+f_u u\Big\}$$

$$= \quad 0$$

with $\hat{y} = y_{t-1} - \bar{y}$, $u = u_t$, $u' = u_{t+1}$, $f_{y_+} = \frac{\partial f}{\partial y_{t+1}}$, $f_{y_0} = \frac{\partial f}{\partial y_t}$, $f_{y_-} = \frac{\partial f}{\partial y_{t-1}}$, $f_u = \frac{\partial f}{\partial u_t}$, $g_y = \frac{\partial g}{\partial y_{t-1}}$, $g_u = \frac{\partial g}{\partial u_t}$, $g_\sigma = \frac{\partial g}{\partial \sigma}$.

Taking the expectation

$$
\begin{aligned}
E_t\left\{F^{(1)}(y_{t-1},u_t,{\color{red}u_{t+1}},\sigma)\right\} = \\
& f(\bar{y},\bar{y},\bar{y},0)+f_{y_+}\left(g_y\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+g_\sigma\sigma\right) \\
& +f_{y_0}\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+f_{y_-}\hat{y}+f_u u\Big\} \\
= & \left(f_{y_+}g_y g_y+f_{y_0}g_y+f_{y_-}\right)\hat{y}+\left(f_{y_+}g_y g_u+f_{y_0}g_u+f_u\right)u \\
& +\left(f_{y_+}g_y g_\sigma+f_{y_0}g_\sigma\right)\sigma \\
= & 0
\end{aligned}
$$

Recovering g_y

$$\left(f_{y_+} g_y g_y + f_{y_0} g_y + f_{y_-}\right) \hat{y} = 0$$

Structural state space representation:

$$\begin{bmatrix} 0 & f_{y_+} \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} = \begin{bmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y}$$

or

$$\begin{bmatrix} 0 & f_{y_+} \\ I & 0 \end{bmatrix} \begin{bmatrix} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{bmatrix} = \begin{bmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{bmatrix} \begin{bmatrix} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{bmatrix}$$

Structural state space representation

$$Dx_{t+1} = Ex_t$$

with

$$x_{t+1} = \begin{bmatrix} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{bmatrix} \qquad x_t = \begin{bmatrix} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{bmatrix}$$

- There is an infinity of solutions but we want a unique stable one.
- Problem when D is singular.

Real generalized Schur decomposition

Taking the real generalized Schur decomposition of the pencil $< E, D >$:

$$\begin{aligned} D &= QTZ \\ E &= QSZ \end{aligned}$$

with T, upper triangular, S quasi-upper triangular, $Q'Q = I$ and $Z'Z = I$.

Generalized eigenvalues

λ_i solves

$$\lambda_i D x_i = E x_i$$

For diagonal blocks on S of dimension 1 x 1:

- $T_{ii} \neq 0$: $\lambda_i = \frac{S_{ii}}{T_{ii}}$
- $T_{ii} = 0$, $S_{ii} > 0$: $\lambda = +\infty$
- $T_{ii} = 0$, $S_{ii} < 0$: $\lambda = -\infty$
- $T_{ii} = 0$, $S_{ii} = 0$: $\lambda \in \mathcal{C}$

Applying the decomposition

$$
\begin{aligned}
D\begin{bmatrix} I \\ g_y \end{bmatrix} g_y\hat{y} &= E\begin{bmatrix} I \\ g_y \end{bmatrix}\hat{y} \\
\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I \\ g_y \end{bmatrix} g_y\hat{y} & \\
&= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I \\ g_y \end{bmatrix}\hat{y}
\end{aligned}
$$

Selecting stable trajectory

To exclude explosive trajectories, one imposes

$$Z_{21} + Z_{22} g_y = 0$$

$$g_y = -Z_{22}^{-1} Z_{21}$$

A unique stable trajectory exists if Z_{22} is non-singular: there are as many roots larger than one in modulus as there are forward–looking variables in the model (Blanchard and Kahn condition) and the rank condition is satisfied.

Recovering g_u

$$f_{y_+} g_y g_u + f_{y_0} g_u + f_u = 0$$

$$g_u = -\left(f_{y_+} g_y + f_{y_0}\right)^{-1} f_u$$

Recovering g_σ

$$f_{y_+} g_y g_\sigma + f_{y_0} g_\sigma = 0$$

$$g_\sigma = 0$$

Yet another manifestation of the certainty equivalence property of first order approximation.

irst order approximated decision function

$$y_t = \bar{y} + g_y \hat{y} + g_u u$$

$$\begin{aligned} E\left\{y_t\right\} &= \bar{y} \\ \Sigma_y &= g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \end{aligned}$$

The variance is solved for with an algorithm for Lyapunov equations.

Second order approximation of the model

$$
\begin{aligned}
& E_t\left\{F^{(2)}(y_{t-1},u_t,u_{t+1},\sigma)\right\} = \\
& \quad E_t\Big\{F^{(1)}(y_{t-1},u_t,u_{t+1},\sigma) \\
& \quad +0.5\Big(F_{y_-y_-}(\hat{y}\otimes\hat{y}) + F_{uu}(u\otimes u) + F_{u'u'}(u'\otimes u') + F_{\sigma\sigma}\sigma^2\Big) \\
& \quad +F_{y_-u}(\hat{y}\otimes u) + F_{y_-u'}(\hat{y}\otimes u') + F_{y_-\sigma}\hat{y}\sigma + F_{uu'}(u\otimes u) + F_{u\sigma}u\sigma + F_{u'\sigma}u'\sigma\Big\} \\
= & \quad E_t\Big\{F^{(1)}(y_{t-1},u_t,u_{t+1},\sigma)\Big\} \\
& \quad +0.5\Big(F_{y_-y_-}(\hat{y}\otimes\hat{y}) + F_{uu}(u\otimes u) + F_{u'u'}(\sigma^2\vec{\Sigma}_\epsilon) + F_{\sigma\sigma}\sigma^2\Big) \\
& \quad +F_{y_-u}(\hat{y}\otimes u) + F_{y_-\sigma}\hat{y}\sigma + F_{u\sigma}u\sigma \\
= & \quad 0
\end{aligned}
$$

Representing the second order derivatives

The second order derivatives of a vector of multivariate functions is a three dimensional object. We use the following notation

$$\frac{\partial^2 F}{\partial x \partial x} = \begin{bmatrix} \frac{\partial^2 F_1}{\partial x_1 \partial x_1} & \frac{\partial^2 F_1}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 F_1}{\partial x_2 \partial x_1} & \dots & \frac{\partial^2 F_1}{\partial x_n \partial x_n} \\ \frac{\partial^2 F_2}{\partial x_1 \partial x_1} & \frac{\partial^2 F_2}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 F_2}{\partial x_2 \partial x_1} & \dots & \frac{\partial^2 F_2}{\partial x_n \partial x_n} \\ \vdots & \vdots \ddots & \vdots & \ddots & \vdots & \\ \frac{\partial^2 F_m}{\partial x_1 \partial x_1} & \frac{\partial^2 F_m}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 F_m}{\partial x_2 \partial x_1} & \dots & \frac{\partial^2 F_m}{\partial x_n \partial x_n} \end{bmatrix}$$

Composition of two functions

Let

$$\begin{aligned} y &= g(s) \\ f(y) &= f(g(s)) \end{aligned}$$

then,

$$\frac{\partial^2 f}{\partial s \partial s} = \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial s \partial s} + \frac{\partial^2 f}{\partial y \partial y}\left(\frac{\partial g}{\partial s} \otimes \frac{\partial g}{\partial s}\right)$$

Recovering g_{yy}

$$\begin{aligned} F_{y_-y_-} &= f_{y_+}\left(g_{yy}(g_y \otimes g_y) + g_y g_{yy}\right) + f_{y_0} g_{yy} + B \\ &= 0 \end{aligned}$$

where B is a term that doesn't contain second order derivatives of $g()$.
The equation can be rearranged:

$$\left(f_{y_+} g_y + f_{y_0}\right) g_{yy} + f_{y_+} g_{yy}(g_y \otimes g_y) = -B$$

This is a Sylvester type of equation and must be solved with an appropriate algorithm.

Recovering g_{yu}

$$\begin{aligned} F_{y_-u} &= f_{y_+}\left(g_{yy}(g_y \otimes g_u) + g_y g_{yu}\right) + f_{y_0} g_{yu} + B \\ &= 0 \end{aligned}$$

where B is a term that doesn't contain second order derivatives of $g()$.
This is a standard linear problem:

$$g_{yu} = -\left(f_{y_+} g_y + f_{y_0}\right)^{-1}\left(B + f_{y_+} g_{yy}(g_y \otimes g_u)\right)$$

Recovering g_{uu}

$$\begin{aligned} F_{uu} &= f_{y_+}\left(g_{yy}(g_u \otimes g_u) + g_y g_{uu}\right) + f_{y_0} g_{uu} + B \\ &= 0 \end{aligned}$$

where B is a term that doesn't contain second order derivatives of $g()$.
This is a standard linear problem:

$$g_{uu} = -\left(f_{y_+} g_y + f_{y_0}\right)^{-1}\left(B + f_{y_+} g_{yy}(g_u \otimes g_u)\right)$$

Recovering $g_{y\sigma}$, $g_{u\sigma}$

$$
\begin{aligned}
F_{y\sigma} &= f_{y_+} g_y g_{y\sigma} + f_{y_0} g_{y\sigma} \\
&= 0 \\
F_{u\sigma} &= f_{y_+} g_y g_{u\sigma} + f_{y_0} g_{u\sigma} \\
&= 0
\end{aligned}
$$

as $g_\sigma = 0$. Then

$$g_{y\sigma} = g_{u\sigma} = 0$$

Recovering $g_{\sigma\sigma}$

$$
\begin{aligned}
F_{\sigma\sigma} + F_{u'u'}\Sigma_\epsilon &= f_{y_+}\left(g_{\sigma\sigma} + g_y g_{\sigma\sigma}\right) + f_{y_0} g_{\sigma\sigma} \\
&\quad + \left(f_{y_+y_+}(g_u \otimes g_u) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon \\
&= 0
\end{aligned}
$$

taking into account $g_\sigma = 0$.
This is a standard linear problem:

$$
g_{\sigma\sigma} = -\left(f_{y_+}(I + g_y) + f_{y_0}\right)^{-1}\left(f_{y_+y_+}(g_u \otimes g_u) + f_{y_+} g_{uu}\right)\vec{\Sigma}_\epsilon
$$

Second order decision functions

$$y_t = \bar{y} + 0.5 g_{\sigma\sigma}\sigma^2 + g_y \hat{y} + g_u u + 0.5\left(g_{yy}(\hat{y}\otimes\hat{y}) + g_{uu}(u \otimes u)\right) + g_{yu}(\hat{y}\otimes u)$$

We can fix $\sigma = 1$.

Second order accurate moments:

$$\begin{aligned}
\Sigma_y &= g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \\
E\{y_t\} &= (I - g_y)^{-1}\left(\bar{y} + 0.5\left(g_{\sigma\sigma} + g_{yy}\vec{\Sigma}_y + g_{uu}\vec{\Sigma}_\epsilon\right)\right)
\end{aligned}$$

Stochastic versus deterministic SS

Deterministic steady state: the point where the agents decide to stay, in the absence of shocks, and ignoring futur shocks.

Stochastic steady state: the point where the agents decide to stay, in the absence of shocks, but taking into account the likelihood of futur shocks.

It is possible to compute a second order approximation around the stochastic steady state.

Further issues

- Impulse response functions depend of state at time of shocks and history of future shocks.
- For large shocks second order approximation simulation may explode
  - pruning algorithm (Sims)
  - truncate normal distribution (Judd)

DYNARE commands

Commands:

- check;
- shocks; . . . end;
- stoch_simul(*options*) *variable list*;

Options:

- order = 1,[2]
- solve_algo = 0,1,[2]
- dr_algo = [0],1
- irf = 0,. . . ,[40],. . .
- noprint

Optimal Linear Regulator

Consider,

$$\max_{\{u\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left(y_t' W_{11} y_t + 2 y_t' W_{12} u_t + u_t' W_{22} u_t \right)$$

s.t.

$$A_+ E_t \left(y_{t+1} \right) + A_0 y_t + A_- y_{t-1} + B u_t + C e_t = 0$$

Lagrangian:

$$\begin{aligned} L \quad = \quad & E_1 \sum_{t=1}^{\infty} \beta^{t-1} \Big[y_t' W_{11} y_t + 2 y_t' W_{12} u_t + u_t' W_{22} u_t \\ & + \lambda_t' \left(A_+ E_t \left(y_{t+1} \right) + A_0 y_t + A_- y_{t-1} + B u_t + C e_t \right) \Big] \end{aligned}$$

First order conditions

$$
\begin{aligned}
\frac{\partial L}{\partial y_1} &= 2W_{11}y_1 + 2W_{12}u_t + A_0'\lambda_1 + \beta A_-' E_1\left(\lambda_2\right) \\
&= 0 \\
\frac{\partial L}{\partial y_t} &= 2W_{11}y_t + 2W_{12}u_t + \beta^{-1}A_+'\lambda_{t-1} + A_0'\lambda_t + \beta A_-' E_t\left(\lambda_{t+1}\right) \qquad t = 2, \ldots \\
&= 0 \\
\frac{\partial L}{\partial u_t} &= 2W_{12}'y_t + 2W_{22}u_t + B'\lambda_t \qquad t = 1, \ldots \\
&= 0 \\
\frac{\partial L}{\partial \lambda_t} &= A_+ E_t\left(y_{t+1}\right) + A_0 y_t + A_- y_{t-1} + Bu_t + Ce_t \\
&= 0
\end{aligned}
$$

One can write the first equation (for $t = 1$) as the second one (for $t > 1$) if and only if $\lambda_0 = 0$.

Augmented model

$$
\begin{aligned}
2W_{11}y_t + 2W_{12}u_t + \beta^{-1}A'_+\lambda_{t-1} + A'_0\lambda_t + \beta A'_- E_t\left(\lambda_{t+1}\right) &= 0 \\
2W'_{12}y_t + 2W_{22}u_t + B'\lambda_t &= 0 \\
A_+ E_t\left(y_{t+1}\right) + A_0 y_t + A_- y_{t-1} + Bu_t + Ce_t &= 0
\end{aligned}
$$

for y_0 given and $\lambda_0 = 0$.

Example: *cgg_olr.mod*

```
var y inf r;
varexo e_y e_inf;

parameters delta sigma alpha kappa;

delta =  0.44;
kappa =  0.18;
alpha =  0.48;
sigma = -0.06;

model(linear);
y  = delta * y(-1) + (1-delta) * y(+1) + sigma *(r - inf(+1)) + e_y;
inf  =   alpha * inf(-1) + (1-alpha) * inf(+1) + kappa*y + e_inf;
end;
```

Example: *cgg_olr.mod* (continued)

```
shocks;
var e_y; stderr 0.63;
var e_inf; stderr 0.4;
end;

olr_inst  r;
optim_weights;
y  1;
inf  1;
end;

olr;
```