Technology Shocks in the New Keynesian Model

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1 The Model

The economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a central bank. During each period $t = 0, 1, 2, ...$, each intermediate goods-producing firm produces a distinct, perishable intermediate good. Hence, intermediate goods may also be indexed by $i \in [0, 1]$, where firm i produces good i. The model features enough symmetry, however, to allow the analysis to focus on the behavior of a representative intermediate goods-producing firm, identified by the generic index i. The activities of each agent, and their implications for the evolution of equilibrium prices and quantities, will now be described in turn.

1.1 The Representative Household

The representative household enters each period $t = 0, 1, 2, ...$ with money M_{t-1} and bonds B_{t-1}. At the beginning of each period, the household receives a lump-sum nominal transfer T_t from the central bank. Next, the household's bonds mature, providing B_{t-1} additional units of money. The household uses some of this money to purchase B_t new bonds at nominal cost B_t/r_t, where r_t denotes the gross nominal interest rate between t and $t + 1$.

During period t, the household supplies $h_t(i)$ units of labor to each intermediate goods-producing firm $i \in [0, 1]$, for a total of

$$h_t = \int_0^1 h_t(i)di$$

during period t. The household gets paid at the nominal wage W_t. The household consumes C_t units of the finished good, purchased at the nominal price P_t from the representative finished goods-producing firm.

At the end of period t, the household receives nominal profits $D_t(i)$ from each intermediate goods-producing firm $i \in [0,1]$, for a total of

$$D_t = \int_0^1 D_t(i)di.$$

The household then carries M_t units of money into period $t+1$, chosen subject to the budget constraint

$$\frac{M_{t-1} + B_{t-1} + T_t + W_t h_t + D_t}{P_t} \geq C_t + \frac{B_t/r_t + M_t}{P_t}. \tag{1}$$

The household's preferences are described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [a_t \ln(C_t) + f_t \ln(M_t/P_t) - (1/\eta)h_t^\eta],$$

where $1 > \beta > 0$ and $\eta \geq 1$. The preference shocks a_t and f_t follow the autoregressive processes

$$\ln(a_t) = (1-\rho_a)\ln(a) + \rho_a \ln(a_{t-1}) + \varepsilon_{at} \tag{2}$$

and

$$\ln(f_t) = (1-\rho_f)\ln(f) + \rho_f \ln(f_{t-1}) + \varepsilon_{ft}, \tag{3}$$

where $1 > \rho_a \geq 0$, $1 > \rho_f \geq 0$, $a > 0$, $f > 0$, and the zero-mean, serially uncorrelated innovations ε_{at} and ε_{ft} are normally distributed with standard deviations σ_a and σ_f. Utility is additively separable in consumption, real balances, and hours worked; as shown by Driscoll (2000) and Ireland (2002), this additively separability is needed to derive a conventional specification for the IS curve that, in particular, does not include hours worked or real money balances as an additional argument. Given this additive separability, the logarithmic specification for preferences over consumption is necessary, as shown by King, Plosser, and Rebelo (1988), for the model to be consistent with balanced growth.

Thus, the household chooses C_t, h_t, B_t, and M_t for all $t = 0, 1, 2, ...$ to maximize its expected utility subject to the budget constraint (1) for all $t = 0, 1, 2, ...$ The first-order conditions for this problem can be written as

$$h_t^{\eta-1} C_t = a_t(W_t/P_t), \tag{4}$$

$$a_t/C_t = \beta r_t E_t[(a_{t+1}/C_{t+1})(P_t/P_{t+1})], \tag{5}$$

$$M_t/P_t = C_t[r_t/(r_t-1)](f_t/a_t) \tag{6}$$

and (1) with equality for all $t = 0, 1, 2, ...$

1.2 The Representative Finished Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative finished goods-producing firm uses $Y_t(i)$ units of each intermediate good $i \in [0, 1]$, purchased at the nominal price $P_t(i)$, to manufacture Y_t units of the finished good according to the constant-returns-to-scale technology described by

$$\left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t} di\right]^{\theta_t/(\theta_t-1)} \geq Y_t,$$

where, as in Smets and Wouters (2002) and Steinsson (2002), θ_t translates into a random shock to the markup of price over marginal cost. Here, this markup shock follows the autoregressive process

$$\ln(\theta_t) = (1 - \rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{7}$$

where $1 > \rho_\theta \geq 0$, $\theta > 1$, and the zero-mean, serially uncorrelated innovation $\varepsilon_{\theta t}$ is normally distributed with standard deviation σ_θ.

Thus, during period t, the finished goods-producing firm chooses $Y_t(i)$ for all $i \in [0, 1]$ to maximize its profits, which are given by

$$P_t \left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t} di\right]^{\theta_t/(\theta_t-1)} - \int_0^1 P_t(i)Y_t(i)di.$$

The first-order conditions for this problem are

$$Y_t(i) = [P_t(i)/P_t]^{-\theta_t} Y_t$$

for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$

Competition drives the finished goods-producing firm's profits to zero in equilibrium. This zero-profit condition implies that

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t} di\right]^{1/(1-\theta_t)}$$

for all $t = 0, 1, 2, ...$

1.3 The Representative Intermediate Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative intermediate goods-producing firm hires $h_t(i)$ units of labor from the representative household to manufacture $Y_t(i)$ units of intermediate good i according to the constant-returns-to-scale technology described by

$$Z_t h_t(i) \geq Y_t(i). \tag{8}$$

The aggregate technology shock Z_t follows a random walk with drift:

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt}, \tag{9}$$

where $z > 1$ and the zero-mean, serially uncorrelated innovation ε_{zt} is normally distributed with standard deviation σ_z.

Since the intermediate goods substitute imperfectly for one another in producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market; during period t, the firm sets the nominal price $P_t(i)$ for its output, subject to the requirement that it satisfy the representative finished goods-producing firm's demand at that price. And, following Rotemberg (1982), the intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price between periods, measured in terms of the finished good and given by

$$\frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 Y_t,$$

where $\phi \geq 0$ governs the magnitude of the price adjustment cost and where $\pi > 1$ denotes the gross steady-state inflation rate.

The cost of price adjustment makes the intermediate goods-producing firm's problem dynamic; it chooses $P_t(i)$ for all $t = 0, 1, 2, ...$ to maximize its total market value, given by

$$(P_0/\Lambda_0)E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t [D_t(i)/P_t],$$

where $\Lambda_t = a_t/C_t$ measures the marginal utility value to the representative household of additional unit of real profits during period t and where

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} Y_t - \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} \left(\frac{W_t}{P_t}\right)\left(\frac{Y_t}{Z_t}\right) - \frac{\phi}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]^2 Y_t \tag{10}$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem are

$$\begin{aligned} 0 \;=\; & (1-\theta_t)\left(\frac{a_t}{C_t}\right)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t}\left(\frac{Y_t}{P_t}\right) \\ & +\theta_t\left(\frac{a_t}{C_t}\right)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t-1}\left(\frac{W_t}{P_t}\right)\left(\frac{Y_t}{Z_t}\right)\left(\frac{1}{P_t}\right) \\ & -\phi\left(\frac{a_t}{C_t}\right)\left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right]\left[\frac{Y_t}{\pi P_{t-1}(i)}\right] \\ & +\beta\phi E_t\left\{\left(\frac{a_{t+1}}{C_{t+1}}\right)\left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right]\left[\frac{Y_{t+1}}{P_t(i)}\right]\left[\frac{P_{t+1}(i)}{\pi P_t(i)}\right]\right\} \end{aligned} \tag{11}$$

and (9) with equality for all $t = 0, 1, 2, ...$

1.4 Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that $Y_t(i) = Y_t$, $h_t(i) = h_t$, $P_t(i) = P_t$, and $D_t(i) = D_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$ In addition, the market-clearing conditions $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$ must hold for all $t = 0, 1, 2, ...$

After imposing these equilibrium conditions, (1)-(11) become

$$Y_t = C_t + \frac{\phi}{2}\left(\frac{P_t}{\pi P_{t-1}} - 1\right)^2 Y_t, \tag{1}$$

$$\ln(a_t) = (1 - \rho_a)\ln(a) + \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{2}$$

$$\ln(f_t) = (1 - \rho_f)\ln(f) + \rho_f \ln(f_{t-1}) + \varepsilon_{ft}, \tag{3}$$

$$h_t^{\eta-1} C_t = a_t(W_t/P_t), \tag{4}$$

$$a_t/C_t = \beta r_t E_t[(a_{t+1}/C_{t+1})(P_t/P_{t+1})], \tag{5}$$

$$M_t/P_t = C_t[r_t/(r_t - 1)](f_t/a_t), \tag{6}$$

$$\ln(\theta_t) = (1 - \rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{7}$$

$$Z_t h_t = Y_t, \tag{8}$$

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \varepsilon_{zt}, \tag{9}$$

$$\frac{D_t}{P_t} = Y_t - \left(\frac{W_t}{P_t}\right) h_t - \frac{\phi}{2}\left(\frac{P_t}{\pi P_{t-1}} - 1\right)^2 Y_t, \tag{10}$$

and

$$\begin{aligned} 0 \;=\; & 1 - \theta_t + \theta_t\left(\frac{W_t}{P_t}\right)\left(\frac{1}{Z_t}\right) - \phi\left(\frac{P_t}{\pi P_{t-1}} - 1\right)\left(\frac{P_t}{\pi P_{t-1}}\right) \\ & + \beta\phi E_t\left[\left(\frac{a_{t+1}}{a_t}\right)\left(\frac{C_t}{C_{t+1}}\right)\left(\frac{P_{t+1}}{\pi P_t} - 1\right)\left(\frac{P_{t+1}}{\pi P_t}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\right] \end{aligned} \tag{11}$$

for all $t = 0, 1, 2, ...$

1.5 Efficient and Inefficient Shocks

A social planner for this economy chooses Q_t and $L_t(i)$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ a_t \ln(Q_t) - (1/\eta)\left[\int_0^1 L_t(i)di\right]^\eta \right\}$$

subject to

$$Z_t \left[\int_0^1 L_t(i)^{(\theta_t-1)/\theta_t} di \right]^{\theta_t/(\theta_t-1)} \geq Q_t$$

for all $t = 0, 1, 2, ...$ The first-order conditions for this problem can be written as

$$\left[\int_0^1 L_t(i) di \right]^{\eta-1} = (a_t/Q_t)Z_t(Q_t/Z_t)^{1/\theta_t} L_t(i)^{-1/\theta_t}$$

and

$$Z_t \left[\int_0^1 L_t(i)^{(\theta_t-1)/\theta_t} di \right]^{\theta_t/(\theta_t-1)} = Q_t$$

for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$

Note that the first of these two optimality conditions implies that the social planner will choose $L_t(i) = L_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, ...,$ since none of the other objects in that expression depends on i. Hence, the two conditions can be combined to yield

$$Q_t = a_t^{1/\eta} Z_t, \tag{12}$$

which reveals that the efficient level of output Q_t increases when a favorable preference shock a_t or technology shock Z_t is realized. By contrast, the efficient level of output does not depend on the realization of the markup shock θ_t. Below, it is shown that all three of these shocks impact on the model's Phillips curve specification. In more traditional analyses of the Phillips curve, such as that contained in Ball and Mankiw (2002), shocks to the Phillips curve are typically sorted out into those that affect the natural rate of unemployment and those that do not. Here, on the other hand, what distinguishes the two types of shocks is that a_t and Z_t are efficient shocks and θ_t is an inefficient shock.

1.6 The Stationary System

Equations (1)-(12) now form a system involving 13 variables: Y_t, C_t, Q_t, h_t, D_t, r_t, W_t, P_t, M_t, a_t, f_t, θ_t, and Z_t. Define $y_t = Y_t/Z_t$, $c_t = C_t/Z_t$, $q_t = Q_t/Z_t$, $d_t = (D_t/P_t)/Z_t$, $w_t = (W_t/P_t)/Z_t$, $\pi_t = P_t/P_{t-1}$, $m_t = (M_t/P_t)/Z_t$, and $z_t = Z_t/Z_{t-1}$.

In terms of these stationary variables, (1)-(12) can be rewritten as

$$y_t = c_t + (\phi/2)(\pi_t/\pi - 1)^2 y_t, \tag{1}$$

$$\ln(a_t) = (1 - \rho_a) \ln(a) + \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{2}$$

$$\ln(f_t) = (1 - \rho_f) \ln(f) + \rho_f \ln(f_{t-1}) + \varepsilon_{ft}, \tag{3}$$

$$h_t^{\eta-1} c_t = a_t w_t, \tag{4}$$

$$a_t/c_t = \beta r_t E_t[(a_{t+1}/c_{t+1})(1/z_{t+1})(1/\pi_{t+1})], \tag{5}$$

$$m_t = c_t[r_t/(r_t - 1)](f_t/a_t), \tag{6}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{7}$$

$$h_t = y_t, \tag{8}$$

$$\ln(z_t) = \ln(z) + \varepsilon_{zt}, \tag{9}$$

$$d_t = y_t - w_t h_t - (\phi/2)(\pi_t/\pi - 1)^2 y_t, \tag{10}$$

$$\begin{aligned} 0 \;=\; & 1 - \theta_t + \theta_t w_t - \phi(\pi_t/\pi - 1)(\pi_t/\pi) \\ & + \beta\phi E_t[(a_{t+1}/a_t)(c_t/c_{t+1})(\pi_{t+1}/\pi - 1)(\pi_{t+1}/\pi)(y_{t+1}/y_t)], \end{aligned} \tag{11}$$

and

$$q_t = a_t^{1/\eta} \tag{12}$$

for all $t = 0, 1, 2, ...$

It is also useful to define the growth rate of output g_t as

$$g_t = Y_t/Y_{t-1}$$

or

$$g_t = (y_t/y_{t-1})z_t \tag{13}$$

and the output gap x_t as

$$x_t = Y_t/Q_t$$

or

$$x_t = y_t/q_t \tag{14}$$

for all $t = 0, 1, 2, ...$

Use (3), (4), (6), (8), (10), and (12) to eliminate f_t, w_t, m_t, h_t, d_t, and q_t. Now the system can be written more compactly as the eight equations

$$y_t = c_t + (\phi/2)(\pi_t/\pi - 1)^2 y_t, \tag{1}$$

$$\ln(a_t) = (1-\rho_a)\ln(a) + \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{2}$$

$$a_t/c_t = \beta r_t E_t[(a_{t+1}/c_{t+1})(1/z_{t+1})(1/\pi_{t+1})], \tag{5}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \varepsilon_{\theta t}, \tag{7}$$

$$\ln(z_t) = \ln(z) + \varepsilon_{zt}, \tag{9}$$

$$\begin{aligned} 0 \;=\; & 1 - \theta_t + \theta_t (c_t/a_t) y_t^{\eta-1} - \phi(\pi_t/\pi - 1)(\pi_t/\pi) \\ & + \beta\phi E_t[(a_{t+1}/a_t)(c_t/c_{t+1})(\pi_{t+1}/\pi - 1)(\pi_{t+1}/\pi)(y_{t+1}/y_t)], \end{aligned} \tag{11}$$

$$g_t = (y_t/y_{t-1})z_t, \tag{13}$$

and

$$x_t = y_t/a_t^{1/\eta} \tag{14}$$

for all $t = 0, 1, 2, ...$

1.7 The Steady State

In the absence of shocks, the economy converges to a steady-state growth path, in which $y_t = y$, $c_t = c$, $r_t = r$, $\pi_t = \pi$, $g_t = g$, $x_t = x$, $a_t = a$, $\theta_t = \theta$, and $z_t = z$. The steady-state values a, θ, and z are determined exogenously by (2), (7), and (9), while the steady-state value π will be determined by the central bank.

The steady-state values r and g are determined by (5) and (13) as

$$r = \pi(z/\beta)$$

and

$$g = z.$$

The steady-state values c, y, and x are determined by (1), (11), and (14) as

$$y = c = \{a[(\theta - 1)/\theta]\}^{1/\eta}$$

and

$$x = [(\theta - 1)/\theta]^{1/\eta}.$$

1.8 The Linearized System

The system consisting of (1), (2), (5), (7), (9), (11), (13) and (14) can be log-linearized around the steady-state to describe how the economy responds to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{r}_t = \ln(r_t/r)$, $\hat{\pi}_t = \ln(\pi_t/\pi)$, $\hat{g}_t = \ln(g_t/g)$, $\hat{x}_t = \ln(x_t/x)$, $\hat{a}_t = \ln(a_t/a)$, $\hat{\theta}_t = \ln(\theta_t/\theta)$, and $\hat{z}_t = \ln(z_t/z)$. A first-order Taylor approximation to (1) reveals that $\hat{c}_t = \hat{y}_t$, allowing $\hat{c}_t$ to be eliminated from the system. First-order approximations to the remaining seven equations then yield

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{2}$$

$$\hat{x}_t = E_t \hat{x}_{t+1} - (\hat{r}_t - E_t \hat{\pi}_{t+1}) + (1 - 1/\eta)(1 - \rho_a)\hat{a}_t, \tag{5}$$

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \varepsilon_{\theta t}, \tag{7}$$

$$\hat{z}_t = \varepsilon_{zt}, \tag{9}$$

$$\phi \hat{\pi}_t = \beta \phi E_t \hat{\pi}_{t+1} + \eta(\theta - 1)\hat{x}_t - \hat{\theta}_t, \tag{11}$$

$$\hat{g}_t = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t, \tag{13}$$

and

$$\hat{x}_t = \hat{y}_t - (1/\eta)\hat{a}_t \tag{14}$$

for all $t = 0, 1, 2, \ldots.$

In this system, (2), (7), and (9) govern the behavior of the exogenous shocks, while (13) and (14) simply define the growth rate of output and the output gap. Meanwhile, (5) takes the form of a forward-looking IS curve, and (11) is a version of the New Keynesian Phillips curve.

Note that although the preference shock a_t and the technology shock z_t do not appear explicitly in the model's Phillips curve, these two efficient shocks enter implicitly through the definition of the output gap variable $\hat{x}_t$. Note, moreover, that in the absence of the cost-push or markup shock $\hat{\theta}_t$, the central bank can perfectly stabilize both the inflation rate and the output gap by adopting a monetary policy that enables the real market rate of interest $\hat{r}_t - E_t\hat{\pi}_{t+1}$ to track the natural rate of interest $(1-1/\eta)(1-\rho_a)\hat{a}_t$. As emphasized by Clarida, Gali, and Gertler (1999), Gali (2002), and Woodford (2002, Ch.5), only the cost-push shock generates a painful trade-off between output-gap and inflation stabilization as competing goals of monetary policy.

1.9 The Central Bank

The central bank conducts monetary policy by adjusting the short-term nominal interest rate according to the modified Taylor (1993) rule

$$\hat{r}_t = \rho_r\hat{r}_{t-1} + \rho_\pi\hat{\pi}_t + \rho_g\hat{g}_t + \rho_x\hat{x}_t + \varepsilon_{rt}. \tag{15}$$

The lagged interest rate $\hat{r}_{t-1}$ is included among the determinants of the current interest rate $\hat{r}_t$ to allow for a gradual adjustment of policy to the shocks that hit the economy. A sufficiently vigorous long-run response of the interest rate to inflation, as measured by $\rho_\pi/(1-\rho_r)$, is required to insure that this policy rule is consistent with the existence of a unique rational expectations equilibrium; for details, see Parkin (1978), McCallum (1981), Kerr and King (1996), and Clarida, Gali, and Gertler (2000). Since it is unclear whether it most appropriate to depict the central bank as responding to output growth—a variable that it can observe—or the output gap—a variable that is more closely linked to the representative household's welfare, both of these measures of real economic activity are included in the interest rate rule. Finally, in (15), the zero-mean, serially uncorrelated innovation ε_{rt} is normally distributed with standard deviation σ_r.

1.10 Equilibrium Conditions

The workings of the model can now be summarized by the system of eight equations

$$\hat{a}_t = \rho_a\hat{a}_{t-1} + \varepsilon_{at}, \tag{2}$$

$$\hat{x}_t = E_t\hat{x}_{t+1} - (\hat{r}_t - E_t\hat{\pi}_{t+1}) + (1-\omega)(1-\rho_a)\hat{a}_t, \tag{5}$$

$$\hat{e}_t = \rho_e\hat{e}_{t-1} + \varepsilon_{et}, \tag{7}$$

$$\hat{z}_t = \varepsilon_{zt}, \tag{9}$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \psi \hat{x}_t - \hat{e}_t, \tag{11}$$

$$\hat{g}_t = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t, \tag{13}$$

$$\hat{x}_t = \hat{y}_t - \omega \hat{a}_t, \tag{14}$$

and

$$\hat{r}_t = \rho_r \hat{r}_{t-1} + \rho_\pi \hat{\pi}_t + \rho_g \hat{g}_t + \rho_x \hat{x}_t + \varepsilon_{rt} \tag{15}$$

in the eight variables $\hat{y}_t$, $\hat{r}_t$, $\hat{\pi}_t$, $\hat{g}_t$, $\hat{x}_t$, $\hat{a}_t$, $\hat{\theta}_t$, and $\hat{z}_t$. To assist in the empirical implementation of the model, the new parameter ω in (5) and (14) has been defined as $\omega = 1/\eta$, the new parameter ψ in (11) has been defined as $\eta(\theta - 1)/\phi$, and the new shock $\hat{e}_t$ in (7) and (11) has been defined as $(1/\phi)\hat{\theta}_t$. In light of this last definition, $\rho_e = \rho_\theta$, and the zero-mean, serially uncorrelated innovation ε_{et} is normally distributed with standard deviation $\sigma_e = (1/\phi)\sigma_\theta$.

Also to assist in the empirical implementation of the model, lagged output gap and inflation terms can be added to the IS and Phillips curve equations, so that (5) and (11) become

$$\hat{x}_t = \alpha_x \hat{x}_{t-1} + (1 - \alpha_x) E_t \hat{x}_{t+1} - (\hat{r}_t - E_t \hat{\pi}_{t+1}) + (1 - \omega)(1 - \rho_a)\hat{a}_t, \tag{5}$$

and

$$\hat{\pi}_t = \beta \alpha_\pi \hat{\pi}_{t-1} + \beta(1 - \alpha_\pi) E_t \hat{\pi}_{t+1} + \psi \hat{x}_t - \hat{e}_t. \tag{11}$$

The original, microfounded specifications can be recovered, of course, simply by setting $\alpha_x = \alpha_\pi = 0$.

2 Solving the Model

Let

$$s_t^0 = \left[\begin{array}{ccccccc} \hat{y}_{t-1} & \hat{r}_{t-1} & \hat{\pi}_{t-1} & \hat{g}_{t-1} & \hat{x}_{t-1} & \hat{\pi}_t & \hat{x}_t \end{array} \right]'$$

and

$$v_t = \left[\begin{array}{cccc} \hat{a}_t & \hat{e}_t & \hat{z}_t & \varepsilon_{rt} \end{array} \right]'.$$

Then (5), (11), and (13)-(15) can be written as

$$A E_t s_{t+1}^0 = B s_t^0 + C v_t, \tag{16}$$

where A and B are 7×7 and C is 7×4. In particular,

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 1 & 1-\alpha_x \\ 0 & 0 & 0 & 0 & \psi & \beta(1-\alpha_\pi) & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\rho_\pi & -\rho_g & -\rho_x & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -\alpha_x & 0 & 1 \\ 0 & 0 & -\beta\alpha_\pi & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \rho_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} -(1-\omega)(1-\rho_a) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Meanwhile, (2), (7), and (9), can be written as

$$v_t = Pv_{t-1} + \varepsilon_t, \tag{17}$$

where

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{zt} & \varepsilon_{rt} \end{bmatrix}'.$$

Equation (16) describes a system of linear expectational difference equations. This system can be solved by uncoupling the unstable and stable components and then solving the unstable component forward and the stable component backward. There are a number of algorithms for working through this process; the approach taken here uses the method outlined by Klein (2000).

Klein's method relies on the complex generalized Schur decomposition, which identifies unitary matrices Q and Z such that

$$QAZ = S$$

and

$$QBZ = T$$

are both upper triangular, where the generalized eigenvalues of A and B can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(A, B) = \{t_{ii}/s_{ii} | i = 1, 2, ..., 7\}.$$

The matrices Q, Z, S, and T can always be arranged so that the generalized eigenvalues appear in ascending order. Note that there are five predetermined variables and two non-predetermined variables in the vector s_t^0. Thus, if five of the generalized eigenvalues in $\lambda(A, B)$ lie inside the unit circle and two of the generalized eigenvalues lie outside the unit circle, the system has a unique solution. If more than two of the generalized eigenvalues in $\lambda(A, B)$ lie outside the unit circle, then the system has no solution. If less than two of the generalized eigenvalues in $\lambda(A, B)$ lie outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980) and Klein (2000).

Assume from now on that there are exactly two generalized eigenvalues that lie outside the unit circle, and partition the matrices Q, Z, S, and T conformably, so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 is 5×7 and Q_2 is 2×7, and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0_{(2\times5)} & S_{22} \end{bmatrix},$$

and

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{(2\times5)} & T_{22} \end{bmatrix},$$

where Z_{11}, S_{11}, and T_{11} are 5×5, Z_{12}, S_{12}, and T_{12} are 5×2, Z_{21} is 2×5, and Z_{22}, S_{22}, and T_{22} are 2×2.

Next, define the vector s_t^1 of auxiliary variables as

$$s_t^1 = Z' s_t^0$$

so that, in particular,

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix},$$

where

$$s_{1t}^1 = Z'_{11} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{g}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + Z'_{21} \begin{bmatrix} \hat{\pi}_t \\ \hat{x}_t \end{bmatrix} \tag{18}$$

is 5×1 and

$$s_{2t}^1 = Z'_{12} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{g}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + Z'_{22} \begin{bmatrix} \hat{\pi}_t \\ \hat{x}_t \end{bmatrix} \tag{19}$$

is 2×1.

Since Z is unitary, $Z'Z = I$ or $Z' = Z^{-1}$ and hence $s_t^0 = Zs_t^1$. Use this fact to rewrite (16) as

$$AZE_t s_{t+1}^1 = BZs_t^1 + Cv_t.$$

Premultiply this version of (16) by Q to obtain

$$SE_t s_{t+1}^1 = Ts_t^1 + QCv_t$$

or, in terms of the matrix partitions,

$$S_{11}E_t s_{1t+1}^1 + S_{12}E_t s_{2t+1}^1 = T_{11}s_{1t}^1 + T_{12}s_{2t}^1 + Q_1Cv_t \tag{20}$$

and

$$S_{22}E_t s_{2t+1}^1 = T_{22}s_{2t}^1 + Q_2Cv_t. \tag{21}$$

Since the generalized eigenvalues corresponding to the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (21) can be solved forward to obtain

$$s_{2t}^1 = -T_{22}^{-1}Rv_t,$$

where the 2×4 matrix R is given by

$$\begin{aligned} vec(R) &= vec\sum_{j=0}^{\infty}(S_{22}T_{22}^{-1})^j Q_2CP^j = \sum_{j=0}^{\infty} vec[(S_{22}T_{22}^{-1})^j Q_2CP^j] \\ &= \sum_{j=0}^{\infty}[P^j \otimes (S_{22}T_{22}^{-1})^j]vec(Q_2C) = \sum_{j=0}^{\infty}[P \otimes (S_{22}T_{22}^{-1})]^j vec(Q_2C) \\ &= [I_{(8\times 8)} - P \otimes (S_{22}T_{22}^{-1})]^{-1} vec(Q_2C). \end{aligned}$$

Use this result, along with (19), to solve for

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{x}_t \end{bmatrix} = -(Z'_{22})^{-1} Z'_{12} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{g}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} - (Z'_{22})^{-1} T_{22}^{-1} R v_t. \tag{22}$$

Since Z is unitary, $Z'Z = I$ or

$$\begin{bmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} I_{(5\times5)} & 0_{(5\times2)} \\ 0_{(2\times5)} & I_{(2\times2)} \end{bmatrix}.$$

Hence, in particular,

$$Z'_{12} Z_{11} + Z'_{22} Z_{21} = 0$$

or

$$-(Z'_{22})^{-1} Z'_{12} = Z_{21} Z_{11}^{-1}$$

and

$$Z'_{12} Z_{12} + Z'_{22} Z_{22} = I$$

or

$$(Z'_{22})^{-1} = Z_{22} + (Z'_{22})^{-1} Z'_{12} Z_{12} = Z_{22} - Z_{21} Z_{11}^{-1} Z_{12},$$

allowing (22) to be written more conveniently as

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{x}_t \end{bmatrix} = M_1 \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{g}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + M_2 v_t, \tag{22}$$

where

$$M_1 = Z_{21} Z_{11}^{-1}$$

and

$$M_2 = -[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R.$$

Equation (18) now provides a solution for s_{1t}^1:

$$s_{1t}^1 = (Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1}) \begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{g}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} - Z'_{21} [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R v_t.$$

Using

$$Z'_{11}Z_{11} + Z'_{21}Z_{21} = I$$

or

$$Z'_{11} + Z'_{21}Z_{21}Z_{11}^{-1} = Z_{11}^{-1}$$

and

$$Z'_{21}[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}] = Z'_{21}Z_{22} - Z'_{21}Z_{21}Z_{11}^{-1}Z_{12} = -Z_{11}^{-1}Z_{12},$$

this last result can be written more conveniently as

$$s_{1t}^1 = Z_{11}^{-1}\begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{g}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + Z_{11}^{-1}Z_{12}T_{22}^{-1}Rv_t.$$

Finally, substitute these results into (20) to obtain the solution

$$\begin{bmatrix} \hat{y}_t \\ \hat{r}_t \\ \hat{\pi}_t \\ \hat{g}_t \\ \hat{x}_t \end{bmatrix} = M_3\begin{bmatrix} \hat{y}_{t-1} \\ \hat{r}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{g}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} + M_4v_t, \tag{23}$$

where

$$M_3 = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}$$

and

$$M_4 = Z_{11}S_{11}^{-1}(T_{11}Z_{11}^{-1}Z_{12}T_{22}^{-1}R + Q_1C + S_{12}T_{22}^{-1}RP - T_{12}T_{22}^{-1}R) - Z_{12}T_{22}^{-1}RP.$$

Note that since the rows of M_1 and M_2 simply reproduce two of the rows of M_3 and M_4, the model's solution can be written compactly by combining (17) and (23) as

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1}, \tag{24}$$

where

$$s_t = \begin{bmatrix} \hat{y}_{t-1} & \hat{r}_{t-1} & \hat{\pi}_{t-1} & \hat{g}_{t-1} & \hat{x}_{t-1} & \hat{a}_t & \hat{e}_t & \hat{z}_t & \varepsilon_{rt} \end{bmatrix}',$$

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{at+1} & \varepsilon_{et+1} & \varepsilon_{zt+1} & \varepsilon_{rt+1} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} M_3 & M_4 \\ 0_{(4\times5)} & P \end{bmatrix},$$

and

$$W = \begin{bmatrix} 0_{(5\times4)} \\ I_{(4\times4)} \end{bmatrix}.$$

3 Estimating the Model

The model has implications for three observable variables: output growth, inflation, and the short-term nominal interest rate. The empirical model has 17 parameters: z, π, β, ω, ψ, α_x, α_π, ρ_r, ρ_π, ρ_g, ρ_x, ρ_a, ρ_e, σ_a, σ_e, σ_z, and σ_r. Note that z, π, and β determine the steady-state values of output growth, inflation, and the short-term nominal interest rate in the model; hence, values for these parameters can be set in order to match the average levels of the same three variables in the data.

To estimate the remaining parameters via maximum likelihood, let $\{d_t\}_{t=1}^T$ denote the series for the logarithmic deviations of output growth, inflation, and the short-term nominal interest rate from their average, or steady-state, values:

$$d_t = \begin{bmatrix} \hat{g}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(Y_t) - \ln(Y_{t-1}) - \ln(g) \\ \ln(P_t) - \ln(P_{t-1}) - \ln(\pi) \\ \ln(r_t) - \ln(r) \end{bmatrix}.$$

Equation (24) then gives rise to an empirical model of the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{25}$$

and

$$d_t = Cs_t, \tag{26}$$

where $A = \Pi$, $B = W$, C is formed from the rows of Π as

$$C = \begin{bmatrix} \Pi_4 \\ \Pi_3 \\ \Pi_2 \end{bmatrix},$$

and the vector of zero-mean, serially uncorrelated innovations ε_{t+1} is normally distributed with diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}^{'} = \begin{bmatrix} \sigma_a & 0 & 0 & 0 \\ 0 & \sigma_e & 0 & 0 \\ 0 & 0 & \sigma_z & 0 \\ 0 & 0 & 0 & \sigma_r \end{bmatrix}.$$

The model defined by (25) and (26) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t | d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (25) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(9\times 1)} \tag{27}$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1 s_1') = [I_{(81\times 81)} - A \otimes A]^{-1} vec(BVB'). \tag{28}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (26) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned} \hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1} u_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1} u_t. \end{aligned}$$

Hence, from (25),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1} u_t.$$

Using this last result, along with (25) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1} u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t | d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_t C'(C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1} C\Sigma_t A',$$

and initial conditions $\hat{s}_1$ and Σ_1 are provided by (27) and (28).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -\left(\frac{3T}{2}\right)\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T} u_t'\Omega_t^{-1}u_t.$$

4 Evaluating the Model

4.1 Variance Decompositions

Begin by considering (25), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or

$$(I - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B \varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BVB' + ABVB'A' + A^2 BVB'A^{2\prime} + ... + A^{k-1}BVB'A^{k-1\prime}.\end{aligned}$$

In addition, (25) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$vec(\Sigma^s) = [I_{(81\times 81)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (26), which implies that

$$\Sigma_k^d = E(d_{t+k} - E_t d_{t+k})(d_{t+k} - E_t d_{t+k})' = C\Sigma_k^s C',$$

and

$$\Sigma^d = \lim_{k\to\infty} \Sigma_k^d = C\Sigma^s C'.$$

Let Θ denote the vector of estimated parameters, and let H denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

Note that the elements of Σ_k^s, Σ^s, Σ_k^d, and Σ^d can all be expressed as nonlinear functions of Θ:

$$\Sigma = g(\Theta),$$

so that asymptotic standard errors for these elements can be found by calculating

$$\nabla g H \nabla g'.$$

In practice, the gradient ∇g can be evaluated numerically, as suggested by Runkle (1987).

4.2 Testing for Parameter Stability

The procedures described by Andrews and Fair (1988) can be used to test for the stability of the model's estimated parameters. Let Θ^1 and Θ^2 denote the estimated parameters from two disjoint subsamples, and let H^1 and H^2 denote the associated covariance matrices, so that asymptotically,

$$\Theta^1 \sim N(\Theta^{10}, H^1)$$

and

$$\Theta^2 \sim N(\Theta^{20}, H^2).$$

One way of testing for the stability of all of the estimated parameters is with the likelihood ratio statistic

$$LR = 2[\ln L(\Theta^1) + \ln L(\Theta^2) - \ln L(\Theta)],$$

where $\ln L(\Theta^1)$, $\ln L(\Theta^2)$, and $\ln L(\Theta)$ are the maximized log likelihood functions for the first subsample, the second subsample, and the entire sample. According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of stability, where q is the number of estimated parameters.

Alternatively, the stability of some or all of the parameters can be tested with the Wald statistic

$$W = g(\Theta^1, \Theta^2)'(G\hat{H}G')^{-1}g(\Theta^1, \Theta^2),$$

when the stability restrictions are written as

$$g(\Theta^1, \Theta^2) = 0$$

and where

$$G = \frac{\partial g(\Theta^1, \Theta^2)}{\partial(\Theta^1, \Theta^2)}$$

and

$$\hat{H} = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}.$$

If Θ_q^1 and Θ_q^2 denote the subsets of Θ^1 and Θ^2 of interest, and if H_q^1 and H_q^2 denote the covariance matrices of Θ_q^1 and Θ_q^2, then this Wald statistic can be written more simply as

$$W = (\Theta_q^1 - \Theta_q^2)'(H_q^1 + H_q^2)^{-1}(\Theta_q^1 - \Theta_q^2).$$

According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of stability, where q is the number of parameters being tested for stability.