Households:

Households have preferences over consumption (c), labor (h), and environmental quality (S): $u(c, h, S)$. Following King-Plosser-Rebelo (KPR) preferences, consumption and leisure, which in this case is a leisure-environment composite, are additively separable with logarithmic utility to ensure that income and substitution effects cancel in equilibrium. The representative household solves the following optimization program

$$\max_{\{c_t,h_t,k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \alpha \log c_t + \frac{(1-\alpha)}{1-\gamma} Q_t^{1-\gamma} \right\}$$

$$\text{where } Q_t = \left((\chi_s)S_t^{-\rho} + (1-\chi_s)(1-h_t)^{\rho}\right)^{1/\rho} \text{ and } S = E\xi$$

subject to

$$c_t(1+\tau_{ct}) + x_t(1+\tau_{xt}) = w_t h_{ct}(1-\tau_{ht}) + (1-\tau_{kt})(r_t-\delta)k_t + \delta k_t + \psi_t$$

Environmental quality is pinned down by an emissions intensity parameter $\xi \in (0,1)$; pollution enters as $(1/S)^{\rho}$ into utility, meaning that higher pollution lowers the quality of the environment to the household. γ is the elasticity of substitution for leisure, and α is a share parameter on the relative utility of consumption versus the leisure-environment composite. ψ is just a lump sum rebate back to households based on tax revenues.

Final goods:

Final goods producers purchase an intermediate energy good (E) and combine it with capital and labor in order to produce a homogeneous final good that households consume. They solve the following optimization problem

$$\max_{K_{1t},H_{1t},E_t} \Pi_1 = \left\{ \exp(z_t) \left[\chi_h (K_t^{\theta} H_t^{1-\theta})^{\nu} + (1-\chi_h)E_t^{\nu}\right]^{1/\nu} - w_t H_t - r_t K_t - p_t E_t \right\}$$

Since markets are perfectly competitive, factor prices are equal to their marginal products

$$r_{1t} = \partial Y/\partial K_1 = \exp(z_t) \left[\chi_h (K_t^{\theta} H_t^{1-\theta})^{\nu} + (1-\chi_h)E_t^{\nu}\right]^{\frac{1-\nu}{\nu}} \chi_h (K_t^{\theta} H_t^{1-\theta})^{\nu-1} \theta K_t^{\theta-1} H_t^{1-\theta}$$

$$w_{1t} = \partial Y/\partial H_1 = \exp(z_t) \left[\chi_h (K_t^{\theta} H_t^{1-\theta})^{\nu} + (1-\chi_h)E_t^{\nu}\right]^{\frac{1-\nu}{\nu}} \chi_h (K_t^{\theta} H_t^{1-\theta})^{\nu-1} (1-\theta) K_t^{\theta} H_t^{-\theta}$$

$$p_t = \partial Y/\partial E = \exp(z_t) \left[\chi_h (K_t^{\theta} H_t^{1-\theta})^{\nu} + (1-\chi_h)E_t^{\nu}\right]^{\frac{1-\nu}{\nu}} (1-\chi_h) E_t^{\nu-1}$$

Using the FOC for the price of energy, together with their production technology, then solve for E as a function of its other variables to yield a demand for energy of the form

$$E_{demand} = Y_t \left[\frac{(1-\chi_h)\left(\chi_h^{\sigma} w_t^{1-\sigma} + \chi_h^{\sigma} r_t^{1-\sigma} + (1-\chi_h)^{\sigma} p_t^{1-\sigma}\right)^{1/(1-\sigma)}}{p_t} \right]^{\sigma}$$

where $\sigma = 1 - (1/\nu)$, or, $\nu = 1/(1-\sigma)$.

Energy sector:

Intermediate energy producers combine capital and labor to produce an energy good that is sold to the final goods sector. They solve the following optimization problem

$$\max_{p_t} \Pi_2 = (p_t - \tau_{dt})E_{t,demand} - \kappa(E_{t,demand})$$

where τ_d is an environmental (energy) tax and κ is the cost of producing energy in terms of the final-good units; I take it to be linear. Although these markets tend to be oligopolistic and monopolistic, perfect competition is an innocuous assumption in my model. Since the profit function is differentiable at p, then the supply of energy is equal to

$$\frac{\partial \Pi_2}{\partial p} = E_{supply}$$

which will be a very non-linear function, that I solve in Matlab.

Underlying these solutions are two assumptions.

Equilibrium conditions:

The equilibrium solution is characterized by the following

$$\{c_t, k_{t+1}, h_t, e_t, S_t, y_t, z_t\}$$

There are 7 unknowns, so 7 conditions are needed to solve for the equilibrium. First, the intertemporal and intratemporal Euler conditions, respectively, are

$$\beta\left(\frac{C_t}{C_{t+1}}\right)[\exp(z_{ct})(1-\delta)+r_{t+1}(1-\tau_{kt+1})/(1+\tau_{xt})]=1$$

$$C_t\frac{(1-\alpha)}{\alpha}(1-\chi_s)(1-H_t)^{\rho-1}\left[(\chi_s)S^{-\rho}+(1-\chi_s)(1-H_t)^{\rho}\right]^{\frac{1-\gamma-\rho}{\rho}}=\frac{(1-\tau_{ht})(1+\tau_{xt})}{(1+\tau_{ct})}w_t$$

giving two more equations, and where r and w are given previously.

Second, final goods production are given by

$$Y_t=\left[\chi_h(K_t^{\theta}H_t^{1-\theta})^{\nu}+(1-\chi_h)E_t^{\nu}\right]^{1/\nu}$$

giving one more equation, and with a price of energy equal to its marginal product

$$p_t=\partial Y/\partial E=\left[\chi_h(K_t^{\theta}H_t^{1-\theta})^{\nu}+(1-\chi_h)E_t^{\nu}\right]^{\frac{1-\nu}{\nu}}(1-\chi_h)E_t^{\nu-1}$$

that can be solved (inverted) to yield a demand for energy

$$E_{demand}=Y_t\left[\frac{(1-\chi_h)\left(\chi_h^{\sigma}w_t^{1-\sigma}+\chi_h^{\sigma}r_t^{1-\sigma}+(1-\chi_h)^{\sigma}p_t^{1-\sigma}\right)^{1/(1-\sigma)}}{p_t}\right]^{\sigma}$$

giving one more equation. The supply of energy is pinned down by intermediate firms solving

$$\max_{p_t}\Pi_2=(p_t-\tau_{dt})E_{t,demand}-\kappa(E_{t,demand})$$

where κ is taken to be linear. The implicit assumption behind the linear specification is that the costs of producing the intermediate goods is identical to the technology involved in the production of the final goods. A useful interpretation is that intermediate goods producers buy final goods and transform them into intermediate goods using their unique blueprints that they hold under monopolistic competition. Because this is an *incredibly* non-linear problem, I use Matlab to compute the derivative numerically; call the resulting equation E_{supply} so that in equilibrium

$$E_{demand}=E_{supply}$$

giving one more equation. These intermediate firms are price takers in the input markets for capital and labor, but monopolistic in the output markets where they set prices for their energy product and take demand as given. Third, pollution and the shocks, respectively, are given by

$$S_t=E_t\xi$$

$$z_t=z_{t-1}\varsigma+\epsilon_t$$

giving two more equations. Sixth, the resource constraint for the economy is

$$Y_t=C_t+K_t-(1-\delta)K_t$$

giving the last equation, which just says that final output needs to equal consumption, plus investment.

rework below

Definition 1. *The **competitive equilibrium** is a set of aggregate quantities for households,* $\{C_t, H_t, K_{t+1}\}$*, intermediate firms,* $\{H_{2t}, K_{2t}\}$*, final goods firms,* $\{H_{1t}, K_{1t}, E_t\}$*, market prices,* $\{r_{1t}, r_{2t}, w_{1t}, w_{2t}, p_t\}$*, and a distribution of shocks,* $\{z_t\}$*, such that households maximize their utilities, firms maximize profits, and markets clear*

1. $\{C_t, H_t, K_{t+1}\}$ *satisfy the household's problem*
2. $\{K_{1t}, H_{1t}, E_t\}$ *and* $\{K_{2t}, H_{2t}\}$ *satisfy the final goods and intermediate firms' problems, respectively*
3. *The labor market clears with labor and leisure normalized to unity*
4. *Factor prices are equal across sectors and there is free entry*

$$r_{1t} = r_{2t}; \quad w_{1t} = w_{2t}$$

$$K_{1t} + K_{2t} = K_t; \quad H_{1t} + H_{2t} = H_t$$

5. *The intermediate goods market clears*

$$E_t = K_{2t}^{\theta} H_{2t}^{1-\theta}$$

6. *The final goods market clears*

$$Y_t = C_t + K_t - (1-\delta)K_{t-1}$$

7. *Shocks are characterized by exogenous shocks* $\{z_t\}$*.*