Dealing with nonlinearities

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Sébastien Villemot

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Introduction

- Which nonlinearities?
  - Nonlinear Phillips curve
  - Asymmetric adjustment costs and asymmetric business cycle (*e.g.* downwardly rigid wages as in Kim and Ruge-Murcia, 2009)
  - Zero-lower bound for nominal interest rates
  - Risk premium
- A dilemma:
  - Methods for perfect foresight models are more accurate than for stochastic models
  - If possible, start with the deterministic version of the model

Outline

1. Perfect foresight models
   - Presentation of the problem
   - Shocks: temporary/permanent, unexpected/pre-announced
   - Occasionally binding constraints
2. Higher order approximation of stochastic models
   - Presentation of the problem
   - Gains and costs from high order local approximations
   - Penalty functions for occasionally binding constraints
3. Extended path

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Perfect foresight models

Example: neoclassical growth model

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$c_t + k_t = A_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1}$$

First order conditions:

$$\begin{aligned} c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma}\left(\alpha A_{t+1} k_t^{\alpha-1} + 1 - \delta\right) \\ c_t + k_t &= A_t k_{t-1}^{\alpha} + (1-\delta)k_{t-1} \end{aligned}$$

Steady state:

$$\begin{aligned} \bar{k} &= \left(\frac{1-\beta(1-\delta)}{\beta\alpha\bar{A}}\right)^{\frac{1}{\alpha-1}} \\ \bar{c} &= \bar{A}\bar{k}^{\alpha} - \delta\bar{k} \end{aligned}$$

The general problem

Deterministic, perfect foresight, case:

$$f(y_{t+1}, y_t, y_{t-1}, u_t) = 0$$

y : vector of endogenous variables

u : vector of exogenous shocks

Note: a model with more than one lead or lag can be put under that form using auxiliary variables.

Solution of deterministic models

- Approximation: impose return to equilibrium in finite time instead of asymptotically
- Useful to study full implications of nonlinearities
- Computes the trajectory of the variables numerically
- Algorithm based on work of Laffargue, Boucekkine and Juillard
- Recently much accelerated by Mihoubi
- Uses a Newton-type method

A two-boundary value problem

Approximation of an infinite horizon model by a finite horizon one

The stacked system for a simulation over T periods:

$$\begin{cases} f(y_2, y_1, y_0, u_1) = 0 \\ f(y_3, y_2, y_1, u_2) = 0 \\ \vdots \\ f(y_{T+1}, y_T, y_{T-1}, u_T) = 0 \end{cases}$$

for y_0 and $y_{T+1} = \bar{y}$ given.

Compact representation:

$$F(Y) = 0$$

where $Y = \begin{bmatrix} y_1' & y_2' & \ldots & y_T' \end{bmatrix}'$.

A Newton approach

- Start from an initial guess $Y^{(0)}$
- Iterate. Updated solutions $Y^{(k+1)}$ are obtained by solving:

$$F(Y^{(k)}) + \left[\frac{\partial F}{\partial Y}\right]\left(Y^{(k+1)} - Y^{(k)}\right) = 0$$

- Terminal condition:

$$||Y^{(k+1)} - Y^{(k)}|| < \varepsilon_Y \text{ and/or } ||F(Y^{(k)})|| < \varepsilon_F$$

A practical difficulty

The size of the Jacobian is very large. For a simulation over T periods of a model with n endogenous variables, it is a matrix of order $n \times T$.

- 15 years ago, it was more of a problem than today: LBJ (the default method in Dynare $\leq$ 4.2) exploited the particular structure of this Jacobian
- Currently, it is possible to handle the Jacobian as one large, sparse, matrix (now the default method in Dynare 4.3)
- Maybe scope for using GPUs

Outline

Example: neoclassical growth model

The social planner problem is as follows:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j},\ell_{t+j})$$

s.t.

$$y_t = c_t + i_t$$
$$y_t = A_t f(k_{t-1},\ell_t)$$
$$k_t = i_t + (1-\delta)k_{t-1}$$
$$A_t = A^\star e^{a_t}$$
$$a_t = \rho\, a_{t-1} + \varepsilon_t$$

where ε_t is an exogenous shock.

Specifications

- Utility function:

$$u(c_t, \ell_t) = \frac{\left[c_t^{\theta}(1-\ell_t)^{1-\theta}\right]^{1-\tau}}{1-\tau}$$

- Production function:

$$f(k_{t-1}, \ell_t) = \left[\alpha k_{t-1}^{\psi} + (1-\alpha)\ell_t^{\psi}\right]^{\frac{1}{\psi}}$$

First order conditions

- Euler equation:

$$u_c(c_t, \ell_t) = \beta\, \mathbb{E}_t \left[u_c(c_{t+1}, \ell_{t+1}) \Big(A_{t+1} f_k(k_t, \ell_{t+1}) + 1 - \delta \Big) \right]$$

- Arbitrage between consumption and leisure:

$$\frac{u_\ell(c_t, \ell_t)}{u_c(c_t, \ell_t)} + A_t f_l(k_{t-1}, \ell_t) = 0$$

- Resource constraint:

$$c_t + k_t = A_t f(k_{t-1}, \ell_t) + (1 - \delta) k_{t-1}$$

Calibration

Weight of consumption in utility	θ	0.357
Risk aversion	τ	2.0
Share of capital in production	α	0.45
Elasticity of substitution capital/labor (fct of...)	ψ	-0.1
Discount factor	β	0.99
Depreciation rate	δ	0.02
Autocorrelation of productivity	ρ	0.8
Steady state level of productivity	$A^{\star}$	1

Scenario 1: Return to equilibrium

Return to equilibrium starting from $k_0 = 0.5\bar{k}$.

Fragment from `rbc_det1.mod`

```
...
steady;

ik = varlist_indices('Capital',M_.endo_names);
CapitalSS = oo_.steady_state(ik);

histval;
Capital(0) = CapitalSS/2;
end;

simul(periods=300);
```

Scenario 2: A temporary shock to TFP

- The economy starts from the steady state
- There is an unexpected negative shock at the beginning of period 1: $\varepsilon_1 = -0.1$

Fragment from rbc_det2.mod

```
...
steady;

shocks;
var EfficiencyInnovation;
periods 1;
values -0.1;
end;

simul(periods=100);
```

Scenario 3: Pre-announced favorable shocks in the future

- The economy starts from the steady state
- There is a sequence of positive shocks to A_t: 4% in period 5 and an additional 1% during the 4 following periods

Fragment from rbc_det3.mod

```
...
steady;

shocks;
var EfficiencyInnovation;
periods 4, 5:8;
values 0.04, 0.01;
end;
```

Scenario 4: A permanent shock

- The economy starts from the initial steady state ($a_0 = 0$)
- In period 1, TFP increases by 5% permanently (and this was unexpected)

Fragment from rbc_det4.mod

```
...
initval;
EfficiencyInnovation = 0;
end;

steady;

endval;
EfficiencyInnovation = (1-rho)*log(1.05);
end;

steady;
```

Scenario 5: A pre-announced permanent shock

- The economy starts from the initial steady state ($a_0 = 0$)
- In period 6, TFP increases by 5% permanently
- A `shocks` block is used to maintain TFP at its initial level during periods 1–5

Fragment from rbc_det5.mod

```
...
// Same initval and endval blocks as in Scenario 4
...

shocks;
var EfficiencyInnovation;
periods 1:5;
values 0;
end;
```

Outline

Zero nominal interest rate lower bound

- Implemented by writing the law of motion under the following form in Dynare:

$$i_t = \max\left\{0, (1-\rho_i)i^* + \rho_i i_{t-1} + \rho_\pi(\pi_t - \pi^*) + \varepsilon^i_t\right\}$$

- *Warning:* this form will be accepted in a stochastic model, but the constraint will not be enforced in that case!

Irreversible investment

Same model than above, but the social planner is constrained to positive investment paths:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j},\ell_{t+j})$$

s.t.

$$\begin{aligned}
y_t &= c_t + i_t \\
y_t &= A_t f(k_{t-1},\ell_t) \\
k_t &= i_t + (1-\delta)k_{t-1} \\
i_t &\geq 0 \\
A_t &= A^\star e^{a_t} \\
a_t &= \rho\, a_{t-1} + \varepsilon_t
\end{aligned}$$

where the technology (f) and the preferences (u) are as above.

First order conditions

$$u_c(c_t,\ell_t) - \mu_t = \beta\mathbb{E}_t\big[u_c(c_{t+1},\ell_{t+1})\,(A_{t+1}f_k(k_t,\ell_{t+1}) + 1 - \delta) - \mu_{t+1}(1-\delta)\big]$$

$$\frac{u_\ell(c_t,\ell_t)}{u_c(c_t,\ell_t)} + A_t f_l(k_{t-1},\ell_t) = 0$$

$$c_t + k_t = A_t f(k_{t-1},\ell_t) + (1-\delta)k_{t-1}$$

$$\mu_t\,(k_t - (1-\delta)k_{t-1}) = 0$$

where $\mu_t \geq 0$ is the Lagrange multiplier associated to the non-negativity constraint for investment.

Writing this model in Dynare

- Writing this model in Dynare using the Lagrange multiplier is not easy
- Right solution: replicate all endogenous variables three times
  - one set of variables satisfying the unconstrained model
  - one set of variables satisfying the model equations when the constraint binds
  - the third set is the realized variables: equal to one or the other of the above set, depending on whether the unconstrained model satisfies the positivity constraint

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Local approximation of stochastic models

The general problem:

$$\mathbb{E}_t f(y_{t+1}, y_t, y_{t-1}, u_t) = 0$$

y : vector of endogenous variables

u : vector of exogenous shocks

with:

$$\begin{aligned} \mathbb{E}(u_t) &= 0 \\ \mathbb{E}(u_t u_t') &= \Sigma_u \\ \mathbb{E}(u_t u_s') &= 0 \text{ for } t \neq s \end{aligned}$$

What is a solution to this problem?

- A solution is a policy function of the form:

$$y_t = g\left(y_{t-1}, u_t, \sigma\right)$$

where σ is the *stochastic scale* of the problem and:

$$u_{t+1} = \sigma\, \varepsilon_{t+1}$$

- The policy function must satisfy:

$$\mathbb{E}_t f\left(g\left(g\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), g\left(y_{t-1}, u_t, \sigma\right), y_{t-1}, u_t\right) = 0$$

Local approximations

$$\hat{g}^{(1)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} + g_y \hat{y}_{t-1} + g_u u_t$$

$$\begin{aligned}\hat{g}^{(2)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} &+ \frac{1}{2} g_{\sigma\sigma} + g_y \hat{y}_{t-1} + g_u u_t \\ &+ \frac{1}{2}\left(g_{yy}\left(\hat{y}_{t-1} \otimes \hat{y}_{t-1}\right) + g_{uu}\left(u_t \otimes u_t\right)\right) \\ &+ g_{yu}\left(\hat{y}_{t-1} \otimes u_t\right)\end{aligned}$$

$$\begin{aligned}\hat{g}^{(3)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} &+ \frac{1}{2} g_{\sigma\sigma} + \frac{1}{6} g_{\sigma\sigma\sigma} + \frac{1}{2} g_{\sigma\sigma y} \hat{y}_{t-1} + \frac{1}{2} g_{\sigma\sigma u} u_t \\ &+ g_y \hat{y}_{t-1} + g_u u_t + \ldots\end{aligned}$$

Outline

Breaking certainty equivalence (1/2)

The combination of future uncertainty (future shocks) and nonlinear relationships makes for precautionary motives or risk premia.

- 1^{st} order: certainty equivalence; today's decisions don't depend on future uncertainty
- 2^{nd} order:

$$\begin{aligned}\hat{g}^{(2)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} &+ \frac{1}{2} g_{\sigma\sigma} + g_y \hat{y}_{t-1} + g_u u_t \\ &+ \frac{1}{2}\left(g_{yy}\left(\hat{y}_{t-1} \otimes \hat{y}_{t-1}\right) + g_{uu}\left(u_t \otimes u_t\right)\right) \\ &+ g_{yu}\left(\hat{y}_{t-1} \otimes u_t\right)\end{aligned}$$

Risk premium is a constant: $\frac{1}{2} g_{\sigma\sigma}$

Breaking certainty equivalence (2/2)

- 3rd order:

$$\hat{g}^{(3)}\left(y_{t+1}, u_t, \sigma\right) = \bar{y} + \frac{1}{2}g_{\sigma\sigma} + \frac{1}{6}g_{\sigma\sigma\sigma} + \frac{1}{2}g_{\sigma\sigma y}\hat{y}_{t-1} + \frac{1}{2}g_{\sigma\sigma u}u_t + g_y\hat{y}_{t-1} + g_u u_t + \dots$$

Risk premium is linear in the state variables:

$$\frac{1}{2}g_{\sigma\sigma} + \frac{1}{6}g_{\sigma\sigma\sigma} + \frac{1}{2}g_{\sigma\sigma y}\hat{y}_{t-1} + \frac{1}{2}g_{\sigma\sigma u}u_t$$

The cost of local approximations

1. High order approximations are accurate around the steady state, and more so than lower order approximations
2. But can be totally wrong far from the steady state (and may be more so than lower order approximations)
3. Error of approximation of a solution $\hat{g}$, at a given point of the state space (y_{t-1}, u_t):

$$\mathcal{E}\left(y_{t-1}, u_t\right) = \mathbb{E}_t f\left(\hat{g}\left(\hat{g}\left(y_{t-1}, u_t, \sigma\right), u_{t+1}, \sigma\right), \hat{g}\left(y_{t-1}, u_t, \sigma\right), y_{t-1}, u_t\right)$$

Outline

Approximation of occasionally binding constraints with penalty functions

The investment positivity constraint is translated into a penalty on the welfare:

$$\max_{\{c_{t+j},\ell_{t+j},k_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} \beta^j u(c_{t+j},\ell_{t+j}) + h \cdot \log(i_{t+j})$$

s.t.

$$y_t = c_t + i_t$$
$$y_t = A_t f(k_{t-1},\ell_t)$$
$$k_t = i_t + (1-\delta)k_{t-1}$$
$$A_t = A^\star e^{a_t}$$
$$a_t = \rho\, a_{t-1} + \varepsilon_t$$

where the technology (f) and the preferences (u) are as before, and h governs the strength of the penalty (*barrier parameter*)

First order conditions

$$u_c(c_t,\ell_t) - \frac{h}{\dot{i}_t} = \beta \mathbb{E}_t \left[u_c(c_{t+1},\ell_{t+1}) \left(A_{t+1} f_k(k_t,\ell_{t+1}) + 1 - \delta \right) - \frac{h}{\dot{i}_{t+1}} \right]$$

$$\frac{u_\ell(c_t,\ell_t)}{u_c(c_t,\ell_t)} + A_t f_l(k_{t-1},\ell_t) = 0$$

$$c_t + k_t = A_t f(k_{t-1},\ell_t) + (1-\delta)k_{t-1}$$

The penalty function approach

- Resemblance with *interior point method* in numerical optimization (aka *barrier method*)
- Preston and Roca (NBER, 2007); Kim, Kollmann and Kim (JEDC, 2010); Cao-Alvira (Comp. Econ., 2010) discuss various specification for the penalty function
- This methods enables one to solve for occasionally binding constraints with local approximations
- *Warning:* fixing problem around the kink may create distortion elsewhere in the state space

Outline

Extended path (EP) algorithm

- Algorithm for creating a stochastic simulated series
- At every period, compute endogenous variables by running a deterministic simulation with:
  - the previous period as initial condition
  - the steady state as terminal condition
  - a random shock drawn for the current period
  - but no shock in the future
- Advantages:
  - shocks are unexpected *at every period*
  - nonlinearities fully taken into account
- Inconvenient: solution under certainty equivalence (Jensen inequality is violated)
- Method introduced by Fair and Taylor (1983)
- Implemented in Dynare 4.3 by Stéphane Adjemian under the command `extended_path`

k-step ahead EP

- Accuracy can be improved by computing conditional expectation by quadrature, computing next period endogenous variables with the previous algorithm
- This is the one-step ahead EP; no more certainty equivalence
- By recurrence, one can compute a k-step ahead EP
- Difficulty: computing complexity grows exponentially with k
- k-step ahead EP currently under development by Stéphane Adjemian

Conclusion

- Variety of approaches available to deal with nonlinearities
- A particular approach needs to be chosen in view of the particular question that is addressed
- Crucial to get a sense for the error of approximation
- Other work in progress:
  - Implementation of global methods (Chebychev colocation, finite elements)
  - Turns out to be difficult to have an "one size fits all" implementation