

## 1 The linearized model

The steady state of the model which consists of seven variables  $w, R, c, y, \pi, m$  and  $g$  is characterized by the following conditions  $w = \frac{\epsilon-1}{\epsilon}$ ,  $\pi = \mu$ ,  $R = \pi/\beta$ ,  $m = c + g$ , and the static equilibrium conditions (5), (6), and (10). The log-linear approximation of the model’s equilibrium conditions around the steady state is given by:

$$\begin{aligned}\widehat{w}_t - \widehat{R}_t &= \left( \sigma_1 \frac{l}{1-l} \frac{c}{c+g} + \sigma \right) \widehat{c}_t + \sigma_1 \frac{l}{1-l} \frac{g}{c+g} \widehat{g}_t, \\ \widehat{c}_t &= \frac{g+c}{c} \widehat{m}_{t+1} - \frac{g}{c} \widehat{g}_t, \\ \sigma E_t \widehat{c}_{t+1} - \sigma \widehat{c}_t &= \widehat{R}_t - E_t \widehat{\pi}_{t+1}, \\ \theta \widehat{w}_t &= -\beta E_t \widehat{\pi}_{t+1} + \widehat{\pi}_t, \\ \widehat{y}_t &= \frac{c}{c+g} \widehat{c}_t + \frac{g}{c+g} \widehat{g}_t, \\ \widehat{g}_t &= \rho \widehat{g}_{t-1} + \varepsilon_t, \\ \widehat{m}_{t+1} - \widehat{m}_t + \widehat{\pi}_t &= 0 \text{ or } \widehat{R}_{t+1} = \rho_\pi E_t \widehat{\pi}_{t+1} + \rho_y E_t \widehat{y}_{t+1}.\end{aligned}$$

## 2 Appendix to proposition 1

**Model solution** The model with exogenous money growth is solved using the method of undetermined coefficients (see McCallum 1999; Uhlig 1999). Given that  $m_t$  is predetermined as of time  $t$  while  $\pi_t$  can jump, the equilibrium law of motion of the endogenous variables should take the following general form:

$$\widehat{\pi}_t = d_1 \widehat{m}_t + d_2 \widehat{g}_t, \quad (29)$$

$$\widehat{m}_{t+1} = d_3 \widehat{m}_t + d_4 \widehat{g}_t, \quad (30)$$

where the  $d_i$ ,  $i \in \{1, 2, 3, 4\}$  coefficients are to be determined. Let  $a_1 \equiv \theta[\sigma(c+g)/c - 1] - \beta > -1$ ,  $b_1 \equiv -1 - \Psi = -1 - \theta[\sigma_1 l/(1-l) + \sigma(c+g)/c] < -1$ , and  $c_1 \equiv -\theta\sigma\rho(g/c) < 0$ . Inserting the solution form (29) and (30) into the model equations (15) and (16) gives  $d_3$  as the stable solution to

$$d_3^2 - \frac{a_1 + b_1}{a_1} d_3 - \frac{1}{a_1} = 0, \quad (31)$$

and

$$d_1 = 1 - d_3, \quad (32)$$

$$d_2 = \frac{c_1}{(-a_1 - b_1) + \{a_1[d_3 + \rho]\}}, \quad (33)$$

$$d_4 = -d_2. \quad (34)$$

For the equilibrium to be unique, one of the roots of (31) must be less than one in absolute value, and one must be larger. In any case, both roots will be real, as in the solution to

(31) the discriminant  $s \equiv \left(\frac{a_1+b_1}{2a_1}\right)^2 + \frac{1}{a_1}$  is clearly positive. This follows from  $\text{sign}\{s\} = \text{sign}\{(a_1+b_1)^2 + 4a_1\}$ , where the right hand side argument  $g(a_1) := (a_1+2)^2 + 4a_1$  has a minimum value of  $4(-b_1-1) > 0$ . Consequently, the discriminant  $s > 0$  and we have two real roots.

Next we want to show that one of the roots is always strictly positive and smaller than one. Consider first the case  $a_1 \in (-1, 0)$ . In that case, both roots must be positive. It can easily be shown in this case that the smaller root is smaller than one,

$$0 < \frac{a_1+b_1}{2a_1} - \sqrt{\left(\frac{a_1+b_1}{2a_1}\right)^2 + \frac{1}{a_1}} < 1,$$

whereas the larger root is unstable. Next assume that  $a_1 > 0$ , implying that the roots can either be negative or positive. It can easily be shown in this case that the larger root is strictly positive and smaller than one. For the smaller root to be outside the unit circle in this case, it is required that  $2a_1 < -b_1 + 1$ . This restriction is fulfilled if we restrict the range of admissible parameters on  $a_1 < 1 \Leftrightarrow \theta < (1+\beta)/(\sigma[c+g]/c-1)$ .

**Solution properties** With  $d_3 \in (0, 1)$  it follows immediately from (32) that  $d_1 \in (0, 1)$ . Further,  $d_2$  clearly has a negative numerator as  $c_1 < 0$ . To find the sign of the denominator, insert the definitions of  $a_1$  and  $b_1$  into the expression for  $d_2$  in (33), to find that the term in round brackets in the denominator is

$$(-a_1 - b_1) = \left(1 + \beta + \theta \left[1 + \frac{\sigma l}{1-l}\right]\right) > 1 + \beta.$$

Since  $d_3 \in (0, 1)$  and, by definition,  $a_1 > -\beta$ , the following inequality for the term in curly brackets in the denominator in (33) holds:

$$\{a_1[d_3 + \rho]\} > -\beta(d_3 + \rho) > -\beta(1 + \rho).$$

Summing up, the denominator cannot be smaller than  $1 - \rho\beta$  and thus, as  $\beta, \rho \in (0, 1)$ , takes a positive sign. Consequently, we have  $d_2 < 0$ . Finally, by (34) this implies that  $d_4 = -d_2 > 0$ .

**Proof of proposition 1** The claims made in the proposition can now easily be seen to be true as, for a shock at period  $s$ , we have that  $\partial\hat{\pi}_s/\partial\hat{g}_s = d_2 < 0$ . Since  $\hat{w}_s = (1/\theta)[\hat{\pi}_s - \beta\hat{\pi}_{s+1}]$  from the Phillips curve (9), it follows that  $\partial\hat{w}_s/\partial\hat{g}_s = [d_2(1 - \beta\rho) - \beta d_4 d_1]/\theta < 0$ . The response of output and real money  $\partial\hat{y}_s/\partial\hat{g}_s = \partial\hat{m}_{s+1}/\partial\hat{g}_s = d_4 > 0$  follows from the equilibrium condition  $\hat{y}_t = \hat{m}_{t+1}$ . The sign of the consumption response is found by inserting the solution into the linearized cash-in-advance constraint, which delivers  $\partial\hat{c}_s/\partial\hat{g}_s = -d_2(c+g)/c - g/c$ ; substituting for  $d_2$  shows that

$$\text{sign}\left\{\frac{\partial\hat{c}_s}{\partial\hat{g}_s}\right\} = \text{sign}\left\{-a_1 d_3 - 1 - \left[\theta\left(1 - \rho + \sigma_1 \frac{l}{1-l}\right) + \beta(1 - \rho)\right]\right\},$$

which is negative as  $d_3 \in (0, 1)$ , given that  $-1 < a_1 < 1$ . Hence,  $\partial\hat{c}_s/\partial\hat{g}_s < 0$  follows. Finally, the response of the nominal interest rate can be derived from the households dynamic first order condition (7), together with (10), the cash-in-advance constraint (3),

and the money growth rule (12) as

$$\partial \widehat{R}_s / \partial \widehat{g}_s = -d_2 \left[ \left( \sigma \frac{c+g}{c} - 1 \right) (1 - d_1) + (1 - \rho) \right] + \sigma(1 - \rho) \left\{ \frac{c+g}{c} d_2 + \frac{g}{c} \right\}.$$

Using that  $d_2 < 0$  and the term in curly brackets equals  $-\partial \widehat{c}_s / \partial \widehat{g}_s$ , we conclude that  $\partial \widehat{R}_s / \partial \widehat{g}_s > 0$  holds. This completes the proof.  $\square$

### 3 Appendix to proposition 2

**Model solution** The interest rate rule model for the case  $\rho_y = 0$  is solved using the method of undetermined coefficients (see McCallum 1999; Uhlig 1999). Given that both  $c$  and  $\pi$  can jump, while  $R$  is predetermined, the general solution for the equilibrium law of motion of the endogenous variables should take the following form

$$\widehat{\pi}_t = d_1 \widehat{g}_t + d_5 \widehat{R}_t \quad (35)$$

$$\widehat{c}_t = d_2 \widehat{g}_t + d_6 \widehat{R}_t \quad (36)$$

$$\widehat{R}_{t+1} = d_3 \widehat{R}_t + d_4 \widehat{g}_t \quad (37)$$

Inserting the solution forms (35) to (37) into the model equations (18) to (20), and setting coefficients to zero, we find that  $d_3(1 - \rho_\pi d_5) = 0$ , hence either  $d_3 = 0$  or  $d_5 = 1/\rho_\pi$ . The first possibility is the relevant one here, since  $d_3 = 0$  is the ‘minimum state variable solution’ in McCallum’s (1983, 1999) sense; it is the only equilibrium that is stable in the sense that the interest rate returns to steady state after a shock. We do not offer a formal proof of uniqueness, but rely on a large literature claiming equilibrium uniqueness in NNS models with active interest rate policy ( $\rho_\pi > 1$  in our notation). Numerical evaluation indeed indicates that the equilibrium is unique, because for any reasonable parameterization we checked there is one stable root (zero) and two unstable roots. Thus, we treat  $d_3 = 0$  as the relevant solution henceforth and conclude the rest of the resulting coefficients to be

$$d_3 = 0, \quad (38)$$

$$d_5 = -\frac{\xi_1}{\sigma} + \theta < 0, \quad (39)$$

$$d_6 = -\frac{1}{\sigma} < 0, \quad (40)$$

$$d_1 = \xi_2(\rho - 1)(\xi_1 \rho_\pi - \sigma \theta \rho_\pi + \sigma) / \Delta, \quad (41)$$

$$d_2 = \xi_2 \rho (\rho_\pi - 1) / \Delta, \quad (42)$$

$$d_4 = \xi_2(\rho - 1) \rho \rho_\pi \sigma / \Delta, \quad (43)$$

where

$$\xi_1 \equiv \theta \frac{\sigma_1 l}{1 - l} \frac{c}{c + g} + \theta \sigma > 0,$$

$$\xi_2 \equiv \theta \frac{\sigma_1 l}{1 - l} \frac{g}{c + g} > 0,$$

$$\Delta \equiv \xi_1(\rho - \rho_\pi) - (\rho - 1)\sigma(\beta \rho + \rho_\pi \theta - 1).$$

**Solution properties** The responses of the endogenous variables to changes in government expenditures depend on the sign of the coefficients  $d_1, d_2$ , and  $d_4$ . Their common denominator can be rewritten as

$$\Delta = -\theta\sigma\rho(\rho_\pi - 1) - (1 - \rho)\sigma(1 - \beta\rho) - \theta\frac{\sigma_1 l}{1 - l}\frac{c}{c + g}(\rho_\pi - \rho) < 0,$$

where the inequality sign follows from active interest rate policy ( $\rho_\pi > 1$ ). The numerator of  $d_1$  in (41) is negative which can immediately be seen by inserting the structural terms for  $\xi_1$  and  $\xi_2$ :

$$-\theta\frac{\sigma_1 l}{1 - l}\frac{g}{c + g}(1 - \rho) \left[ \theta\frac{\sigma_1 l}{1 - l}\frac{c}{c + g}\rho_\pi + \sigma \right] < 0.$$

Hence,  $d_1 > 0$ . Likewise, it is easily seen from (42) and (43) that  $d_2 < 0$  and  $d_4 > 0$ .

**Proof of proposition 2** The claims made in the proposition can now be established, because for a shock to government expenditure in period  $s$  we have  $\partial\widehat{\pi}_s/\partial\widehat{g}_s = d_1 > 0$ ,  $\partial\widehat{c}_s/\partial\widehat{g}_s = d_2 < 0$ , and  $\partial\widehat{R}_{s+1}/\partial\widehat{g}_s = d_4 > 0$ . Since  $\widehat{w}_s = (1/\theta)[\widehat{\pi}_s - \beta\widehat{\pi}_{s+1}]$ , it follows that  $\partial\widehat{w}_s/\partial\widehat{g}_s = [d_1(1 - \beta\rho) - \beta d_4 d_5]/\theta > 0$  as  $d_5 < 0$  from (39). The claim about the output response can be established as follows. Given the coefficients from above, we have by inserting into the aggregate resource constraint that  $\partial\widehat{y}_s/\partial\widehat{g}_s = \left(\frac{c}{c+g}d_2 + \frac{g}{c+g}\right) =: d_7$ . Rearranging and using the results from above,  $d_7$  is found to be

$$d_7 = \frac{\frac{g}{c+g}\theta\frac{\sigma_1 l}{1-l}\frac{c}{c+g}\rho_\pi(1 - \rho) + \frac{g}{c+g}\theta\sigma\rho(\rho_\pi - 1) + \frac{g}{c+g}(1 - \rho)\sigma(1 - \beta\rho)}{\theta\sigma\rho(\rho_\pi - 1) + (1 - \rho)\sigma(1 - \beta\rho) + \theta\frac{\sigma_1 l}{1-l}\frac{c}{c+g}(\rho_\pi - \rho)} > 0$$

because it exclusively consists of positive terms, given that  $\rho_\pi > 1$ . Hence, output rises on impact in response to a positive government spending shock, as claimed in the proposition. Finally, as  $\widehat{y}_s = \widehat{m}_{s+1}$ , we have  $\partial\widehat{m}_{s+1}/\partial\widehat{g}_s > 0$ . This completes the proof.  $\square$

## 4 Appendix to proposition 3

**Model solution** Applying the same approach as above in appendix 3, but for the general case  $\rho_y > 0$ , we find that the solutions for  $d_3, d_5$ , and  $d_6$  are unchanged and given by (38) to (40), while the remaining coefficients are more complex than before.

$$d_1 = \Psi_1/\Delta, \tag{44}$$

$$d_2 = \Psi_2/\Delta, \tag{45}$$

$$d_4 = \Psi_4/\Delta, \tag{46}$$

where

$$\Psi_1 \equiv \xi_2\sigma\rho_y\frac{c}{c+g} [1 - \rho(1 + \beta(1 - \rho) + \theta)] - \sigma^2\theta\rho\rho_y\frac{g}{c+g} + \xi_2\sigma^2(1 - \rho) \left( \xi_2\rho_\pi\frac{c}{c+g} + \sigma \right),$$

$$\Psi_2 \equiv \rho \left[ \xi_2 \left( \frac{c}{c+g}\rho_y - \rho_\pi\sigma + \sigma \right) - \frac{g}{c+g}\rho_y(\xi_1 - \sigma(\beta\rho + \theta - 1)) \right],$$

$$\Psi_4 \equiv \rho \left[ \left( \xi_2\frac{c}{c+g} - \xi_1\frac{g}{c+g} \right) \rho\rho_y + (\rho - 1)\sigma \left( \frac{g}{c+g}(\beta\rho - 1)\rho_y - \xi_2\rho_\pi \right) \right],$$

$$\xi_1 \equiv \theta \frac{\sigma_1 l}{1-l} \frac{c}{c+g} + \theta \sigma > 0,$$

$$\xi_2 \equiv \theta \frac{\sigma_1 l}{1-l} \frac{g}{c+g} > 0,$$

$$\Delta \equiv \sigma \left[ \xi_1 (\rho_\pi - \rho) + (\rho - 1) \sigma (\beta \rho + \rho_\pi \theta - 1) - \frac{c}{c+g} \rho_y (\rho \beta + \rho \theta - 1) \right].$$

**Solution properties** Consider first  $\Psi_4$ , the numerator of  $d_4$  in (46), which is positive if

$$\Psi_4 > 0 \Leftrightarrow \rho_y \{ \theta \rho - (1 - \rho)(1 - \beta \rho) \} < (1 - \rho) \theta \frac{\sigma_1 l}{1-l} \rho_\pi. \quad (47)$$

In the case that the term in curly brackets is negative, this inequality is automatically fulfilled regardless of the value that  $\rho_y$  takes. For the general case, the restriction on  $\rho_y$  that is necessary to ensure that  $\Psi_4 > 0$  can be read off (47). However, rather than working with the necessary condition, it proves more useful to state a sufficient condition that will appear again below and thus simplifies the derivations. Therefore, we restrict the applicability of proposition 3 to values of  $\rho_y$  that are sufficiently small to ensure

$$\rho_y < \Xi \equiv \frac{1 - \rho}{\rho} \sigma_1 \frac{l}{1-l} \rho_\pi. \quad (48)$$

Hence,  $\rho_y < \Xi \Rightarrow \Psi_4 > 0$ .

The numerator of  $d_2$  in (45) is clearly negative as it can be summarized as

$$\Psi_2 = \rho \frac{g}{c+g} \sigma \left[ -\theta \frac{\sigma_1 l}{1-l} (\rho_\pi - 1) - \rho_y (1 - \beta \rho) \right] < 0.$$

The common denominator  $\Delta$  can be shown to be positive for small  $\rho_y$ . While the necessary condition is somewhat complex, the condition (48) used above in the discussion of  $\Psi_4$  is again sufficient, so that  $\rho_y < \Xi \Rightarrow \Delta > 0$ . Hence,  $\rho_y < \Xi \Rightarrow d_2 < 0$ ,  $d_4 > 0$  follows.

The coefficient  $d_1$  in (44) that gives the inflation response cannot be negative for

$$\rho_y < \Gamma \equiv \frac{\rho_\pi \sigma_1 \frac{l}{1-l} \theta (1 - \rho) + \frac{c+g}{c} (1 - \rho) \sigma}{\rho \theta + \rho + \rho \beta (1 - \rho) + \sigma \rho \left( \frac{\sigma_1 l}{1-l} \right)^{-1} \frac{c+g}{c} - 1}, \quad (49)$$

provided the denominator on the right hand side is positive. Unfortunately, this cannot be related to the parameter restriction in (48) that was found to be sufficient to determine the signs of  $d_2$ , and  $d_4$  above.

**Proof of proposition 3** The claims made in the proposition can now be established that for a shock to government expenditure in period  $s$ , if (but not only if)  $\rho_y < \Xi$  is fulfilled, we have  $\partial \hat{c}_s / \partial \hat{g}_s = d_2 < 0$ ,  $\partial \hat{R}_{s+1} / \partial \hat{g}_s = d_4 > 0$ . The claim about the output response can be established as follows. Given the coefficients from above, we have by inserting into the aggregate resource constraint  $\partial \hat{y}_s / \partial \hat{g}_s = \left( \frac{c}{c+g} d_2 + \frac{g}{c+g} \right) =: d_7$ . Writing this coefficient as  $d_7 = \Psi_7 / \Delta$ , where

$$\Psi_7 \equiv \frac{g}{c+g} \sigma \left( \frac{\frac{c}{c+g}}{\left[ \theta \frac{l}{1-l} \sigma_1 \rho_\pi (1-\rho) - \rho_y \rho \theta \right]} + \frac{\frac{c}{c+g} \rho_y (1-\beta\rho)(1-\rho) + (1-\rho)\sigma(1-\beta\rho) + \theta\sigma\rho(\rho_\pi - 1)}{\left[ \theta \frac{l}{1-l} \sigma_1 \rho_\pi (1-\rho) - \rho_y \rho \theta \right]} \right),$$

it is clear from inspection of the term in square brackets that  $\rho_y < \Xi$  is sufficient to ensure  $\Psi_7 > 0$ , so that  $\rho_y < \Xi \Rightarrow \partial \hat{y}_s / \partial \hat{g}_s = d_7 > 0$  follows. As  $\hat{y}_s = \hat{m}_{s+1}$ , we also have  $\rho_y < \Xi \Rightarrow \partial \hat{m}_{s+1} / \partial \hat{g}_s > 0$ . Finally, if  $\rho_y < \Gamma$  is fulfilled,  $\partial \hat{\pi}_s / \partial \hat{g}_s = d_1 > 0$ ; since  $\hat{w}_s = (1/\theta) [\hat{\pi}_s - \beta \hat{\pi}_{s+1}]$ , it follows that in this case also  $\partial \hat{w}_s / \partial \hat{g}_s = [d_1(1-\beta\rho) - \beta d_4 d_5] / \theta > 0$  because  $d_5 < 0$  by (39). This completes the proof.  $\square$

## 5 The linearized model with capital

The steady state of the model with capital in section 4 consists of 10 endogenous variables  $(c, l, w, R, m, e, k, q, mc, r)$  satisfying  $e/k = \delta$ ,  $q = 1$ ,  $1/\beta = r^k + 1 - \delta$ ,  $m = c + g$ ,  $y = c + g + e = l^{1-\alpha} k^\alpha$ ,  $\pi = \mu$ , and the static equilibrium conditions (5), (6), (26), and (27). The log-linear approximation of the model's equilibrium conditions around the steady state is given by:

$$\begin{aligned} \hat{w}_t &= \sigma_1 \frac{l}{1-l} \hat{l}_t + \sigma \hat{c}_t + \hat{R}_t, \\ \hat{c}_t &= \frac{c+g}{c} \hat{m}_{t+1} - \frac{g}{c} \hat{g}_t, \\ \sigma E_t \hat{c}_{t+1} &= \hat{R}_t - E_t \hat{\pi}_{t+1} + \sigma \hat{c}_t, \\ \frac{\Phi''}{\Phi'} \delta \hat{e}_t &= \frac{\Phi''}{\Phi'} \delta \hat{k}_t - \hat{q}_t, \\ -\hat{q}_t &= E_t \left\{ \beta \Phi'' \delta^2 \hat{e}_{t+1} - [1 - \beta(1-\delta)] r^k \hat{k}_{t+1} - \beta \Phi'' \delta^2 \hat{k}_{t+1} + \hat{R}_{t+1} - \hat{\pi}_{t+1} - \beta(1-\delta) \hat{q}_{t+1} \right\}, \\ \hat{k}_{t+1} &= (1-\delta) \hat{k}_t + \delta \hat{e}_t, \\ \hat{\pi}_t &= \theta \hat{m} c_t + \beta E_t \hat{\pi}_{t+1}, \\ \hat{w}_t &= \hat{m} c_t - \alpha \hat{l}_t + \alpha \hat{k}_t, \\ \hat{r}^k_t &= \hat{m} c_t + (1-\alpha) \hat{l}_t - (1-\alpha) \hat{k}_t, \\ \frac{g}{y} \hat{g}_t &= -\frac{c}{y} \hat{c}_t - \frac{e}{y} \hat{e}_t + (1-\alpha) \hat{l}_t + \alpha \hat{k}_t, \\ \hat{g}_t &= \rho \hat{g}_{t-1} + \varepsilon_t, \\ \hat{m}_{t+1} &= \hat{m}_t - \hat{\pi}_t \text{ or } \hat{R}_{t+1} = \rho_R \hat{R}_t + \rho_\pi E_t \hat{\pi}_{t+1} + \rho_y E_t \hat{y}_{t+1}. \end{aligned}$$