Handout Class 2

Advanced Macroeconomics, Fall 2011 - Alessandro Di Nola

October 4, 2011

1 The Economy

We consider an economy with a single good, that can be either consumed or saved in the form of productive phisical capital. Private investment and savings coincide since we are in a closed economy without government. Thus we have:

$$\begin{aligned} c_t + s_t &= y_t, \\ s_t &= i_t, \\ k_{t+1} &= (1-\delta)k_t + i_t. \end{aligned}$$

Preferences of the representative consumer in the economy can be represented through a constant relative risk aversion utility function,

$$U(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0.$$

We also assume that the technology for the production of the single good in the economy, $y_t = z_t k_t^\alpha$, is stochastic, due to the presence of an exogenous stochastic productivity factor z_t. The law of motion of z_t is described by:

$$\log(z_t) = \rho \log(z_{t-1}) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2).$$

2 The planner's problem

In this model there are no externalities of any kind , hence the two welfare theorems apply, and the solution to the planner's problem lead to the same allocation of resources as the competitive equilibrium. Since solving a social planner's problem is much simpler than solving a decentralized economy, we focus on the planner's solution.

The benevolent planner chooses sequences of consumption and physical capital to solve the problem,

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t),$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= (1-\delta)k_t + z_t k_t^{\alpha}, \qquad (1) \\ \log(z_t) &= \rho \log(z_{t-1}) + \varepsilon_t, \\ & \text{given } z_0, k_0. \end{aligned}$$

Notice that here k_t stands for beginning-of-period t stock, chosen in $t-1$, so it is predetermined as of time t. The solution to the above problem is a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ that satisfies the following optimality conditions:

$$c_t + k_{t+1} = (1-\delta)k_t + z_t k_t^{\alpha}, \qquad (2)$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right], \qquad (3)$$

$$\log(z_t) = \rho \log(z_{t-1}) + \varepsilon_t, \text{ given } z_0. \qquad (4)$$

Exercise 1 *Derive the above conditions using the Lagrangian approach. [*Hint*: you should attach a time-varying multiplier $\beta^t \lambda_t$ to each constraint (1)].*

Solution The Lagrangian is

$$L = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t \left[(1-\delta)k_t + z_t k_t^{\alpha} - c_t - k_{t+1} \right]$$

$$\begin{aligned} L = \; & \ldots + \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \beta^t \lambda_t \left[(1-\delta)k_t + z_t k_t^{\alpha} - c_t + k_{t+1} \right] + \\ & + \beta^{t+1} \frac{c_{t+1}^{1-\sigma} - 1}{1-\sigma} + \beta^{t+1} \lambda_{t+1} \left[(1-\delta)k_{t+1} + z_{t+1} k_{t+1}^{\alpha} - c_{t+1} + k_{t+2} \right] + \ldots \end{aligned}$$

Hence the first-order conditions with respect to date-t choice variables c_t and k_{t+1} are:

$$\frac{\partial L}{\partial c_t} = 0 \implies c_t^{-\sigma} = \lambda_t$$

$$\frac{\partial L}{\partial k_{t+1}} = 0 \implies -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} [(1-\delta) + \alpha z_{t+1} k_{t+1}^{\alpha-1}] = 0$$

Rearranging the above conditions yields (3).

Since the equilibrium equations (2)-(4) are nonlinear and do not have a closed-form solution, we can get an approximated solution by following these steps:

1. Calculate the deterministic steady-state, i.e. the a point in the system where all variables are constant over time.

2. Log-linearize the equilibrium conditions around the steady-state.

3. Apply the methods developed by Blanchard-Kahn (1980) or Uhlig (1999) to solve the loglinearized system of stochastic difference equations.

3 The steady-state

In the absence of technology shocks, i.e. $\varepsilon_t = 0$ for all t, the economy converges to a steady-state in which each of the three stationary variables c_t k_t and z_t is constant, i.e. $c_t = \overline{c}$, $k_t = \overline{k}$ and $z_t = \overline{z}$ The steady-state values can be easily computed dropping all expectations and time indexes from (2)-(4):

$$\overline{c} + \overline{k} = (1-\delta)\overline{k} + \overline{z}\overline{k}^{\alpha}, \tag{5}$$

$$\overline{c}^{-\sigma} = \beta \left[\overline{c}^{-\sigma} \left(\alpha \overline{z}\overline{k}^{\alpha-1} + 1 - \delta \right) \right], \tag{6}$$

$$\log(\overline{z}) = \rho \log(\overline{z}). \tag{7}$$

Equations (5)-(7) implicitely define the steady-state values for consumption, capital and technology. From the last equation we get $\overline{z} = 1$. We can obtain a closed form solution for the capital stock from (6):

$$\frac{1}{\beta} = \alpha \overline{z}\overline{k}^{\alpha-1} + 1 - \delta$$

$$\Longrightarrow \overline{k} = \left(\frac{\alpha \overline{z}}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}}.$$

Solving (5) for steady-state consumption delivers:

$$\overline{c} = \overline{z}\overline{k}^{\alpha} - \delta \overline{k}.$$

4 The log-linear approximation

We now proceed to describe how to compute a log-linear approximation of equations (2)-(4). Equations (2)-(4) are a nonlinear difference equation system that does not admit an analytical solution[1]. Therefore we have to take a first-order Taylor approximation around the steady-state. It is more

[1] An analytical solution exists only in a few cases, when for instance utility is logarithmic ($\sigma = 1$) and there is full depreciation ($\delta = 1$). See Pset 1 for an example.

convenient, however, to express all variables as percentage (or log-linear) deviations around steady-state, i.e. $\widehat{x}_t \equiv \frac{x_t - \overline{x}}{\overline{x}} \simeq \log\left(\frac{x_t}{\overline{x}}\right)$, rather than $dx_t = x_t - \overline{x}$. Very briefly[2], the cookbook procedure for log-linearizing is:

1. Take logs
2. Do a first order Taylor expansion about the steady state
3. Simplify (also using the ss relationships) so that everything is expressed in percentage deviations from steady state.

4.1 Log-linearize the resource constraint

Equation (2) is the economy's resource constraint that we reproduce here for convenience:

$$c_t + k_{t+1} = (1-\delta)k_t + z_t k_t^{\alpha}$$

$$\log(c_t + k_{t+1}) = \log[(1-\delta)k_t + z_t k_t^{\alpha}]$$

$$\frac{1}{\overline{c}+\overline{k}}[dc_t + dk_{t+1}] = \frac{1}{(1-\delta)\overline{k} + \overline{z}\overline{k}^{\alpha}}\left[(1-\delta)dk_t + \overline{k}^{\alpha}dz_t + \alpha\overline{z}\overline{k}^{\alpha-1}dk_t\right]$$

$$dc_t + dk_{t+1} = (1-\delta)dk_t + \overline{k}^{\alpha}dz_t + \alpha\overline{z}\overline{k}^{\alpha-1}dk_t$$

$$\overline{c}\frac{dc_t}{\overline{c}} + \overline{k}\frac{dk_{t+1}}{\overline{k}} = (1-\delta)\overline{k}\frac{dk_t}{\overline{k}} + \overline{k}^{\alpha}\overline{z}\frac{dz_t}{\overline{z}} + \alpha\overline{z}\overline{k}^{\alpha-1}\overline{k}\frac{dk_t}{\overline{k}}$$

$$\overline{c}\widehat{c}_t + \overline{k}\widehat{k}_{t+1} = (1-\delta)\overline{k}\widehat{k}_t + \overline{z}\overline{k}^{\alpha}\widehat{z}_t + \alpha\overline{z}\overline{k}^{\alpha}\widehat{k}_t$$

$$\frac{\overline{c}}{\overline{k}}\widehat{c}_t + \widehat{k}_{t+1} = (1-\delta)\widehat{k}_t + \overline{z}\overline{k}^{\alpha-1}\widehat{z}_t + \alpha\overline{z}\overline{k}^{\alpha-1}\widehat{k}_t$$

$$\frac{\overline{c}}{\overline{k}}\widehat{c}_t + \widehat{k}_{t+1} = [(1-\delta) + \alpha\overline{z}\overline{k}^{\alpha-1}]\widehat{k}_t + \overline{z}\overline{k}^{\alpha-1}\widehat{z}_t$$

Using the steady relationship $\alpha\overline{z}\overline{k}^{\alpha-1} + 1 - \delta = \frac{1}{\beta}$ we finally get:

$$\frac{c}{k}\widehat{c}_t + \widehat{k}_{t+1} = \frac{1}{\beta}\widehat{k}_t + \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\widehat{z}_t \tag{8}$$

[2] See the Appendix or Malley (2004) for a more detailed discussion about different log-linearization techniques.

4.2 Loglinearize the Euler equation

Equation (3) is the Euler equation (intertemporal optimality condition):

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \right]$$

$$\begin{aligned}
-\sigma \log c_t &= \log \beta - \sigma \log c_{t+1} + \log \left(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta \right) \\
-\sigma \frac{1}{\overline{c}} dc_t &= -\sigma \frac{1}{\overline{c}} dc_{t+1} + \frac{1}{\alpha \overline{z}\overline{k}^{\alpha-1} + 1 - \delta} \left[\alpha \overline{k}^{\alpha-1} dz_{t+1} + \alpha(\alpha-1)\overline{z}\overline{k}^{\alpha-2} dk_{t+1} \right] \\
-\sigma \widehat{c}_t &= -\sigma \widehat{c}_{t+1} + \beta \left[\alpha \overline{z}\overline{k}^{\alpha-1} \widehat{z}_{t+1} + \alpha(\alpha-1)\overline{z}\overline{k}^{\alpha-1} \widehat{k}_{t+1} \right] \\
-\sigma \widehat{c}_t &= -\sigma \widehat{c}_{t+1} + \beta\alpha\overline{z}\overline{k}^{\alpha-1} \widehat{z}_{t+1} + (\alpha-1)\beta\alpha\overline{z}\overline{k}^{\alpha-1} \widehat{k}_{t+1} \\
-\sigma \widehat{c}_t &= -\sigma \widehat{c}_{t+1} + \left[1 - \beta(1-\delta)\right] \rho \widehat{z}_t + \left[1 - \beta(1-\delta)\right] (\alpha-1) \widehat{k}_{t+1} \qquad (9)
\end{aligned}$$

where we used the steady-state relation $1 = \beta[\alpha\overline{z}\overline{k}^{\alpha-1} + 1 - \delta]$ and $E_t \widehat{z}_{t+1} = \rho \widehat{z}_t$.

4.3 Loglinearize the stochastic process for technology

Equation (4) is the stochastic process for technology:

$$\log(z_t) = \rho \log(z_{t-1}) + \varepsilon_t$$

$$\begin{aligned}
\log(z_t) - \log \overline{z} &= \rho \left[\log(z_{t-1}) - \log \overline{z} \right] + \varepsilon_t \\
\log \left(\frac{z_t}{\overline{z}} \right) &= \rho \log \left(\frac{z_{t-1}}{\overline{z}} \right) + \varepsilon_t \\
\widehat{z}_t &= \rho \widehat{z}_{t-1} + \varepsilon_{t+1}
\end{aligned}$$

5 The Blanchard-Kahn solution method

The equilibrium equations in log-linear form are:

$$\begin{aligned}
\frac{\overline{c}}{\overline{k}} \widehat{c}_t + \widehat{k}_{t+1} &= \frac{1}{\beta} \widehat{k}_t + \frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta \right) \widehat{z}_t, & (10) \\
-\sigma \widehat{c}_t + \sigma E_t \widehat{c}_{t+1} &= \left[1 - \beta(1-\delta)\right] (\alpha-1) \widehat{k}_{t+1} + \left[1 - \beta(1-\delta)\right] \widehat{z}_{t+1}, & (11) \\
\widehat{z}_{t+1} &= \rho \widehat{z}_t + \varepsilon_{t+1}. & (12)
\end{aligned}$$

The above equations can be written in matrix form as:

$$A E_t s_{t+1}^0 = B s_t^0 + C \widehat{z}_t \qquad (13)$$

where $s_t^0 \equiv \begin{bmatrix} \widehat{k}_t \\ \widehat{c}_t \end{bmatrix}$. Therefore

$$\begin{bmatrix} \left(\frac{1}{\beta}-1+\delta\right)(1-\alpha) & \frac{1}{\beta}\sigma \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \widehat{k}_{t+1} \\ E_t\widehat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\beta}\sigma \\ \frac{1}{\beta} & -\frac{c}{k} \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{c}_t \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{\beta}-1+\delta\right)\rho \\ \left(\frac{1}{\beta}-1+\delta\right)\frac{1}{\alpha} \end{bmatrix} \widehat{z}_t. \tag{14}$$

Premultiplying both sides of (13) by A^{-1} yields

$$E_t s_{t+1}^0 = D s_t^0 + F\widehat{z}_t, \tag{15}$$

where $D = A^{-1}B$ and $F = A^{-1}C$.

Claim 2 *The D matrix has one stable eigenvalue and one unstable: $|\mu_1| < 1$, $|\mu_2| > 1/\beta$.*

Proof. It can be shown that the characteristic equation of matrix D is equal to:

$$\mu^2 - \left(1 + \frac{1}{\beta} + A\right)\mu + \frac{1}{\beta} = 0.$$

where $A > 0$ is function of a bunch of parameters. The two roots of the characteristic equation (which are the two eigenvalues of D) satisfy

$$\begin{aligned} \mu_1 + \mu_2 &= 1 + \frac{1}{\beta} + A, \\ \mu_1\mu_2 &= \frac{1}{\beta}, \end{aligned}$$

so that,

$$\mu_1 + \frac{1}{\beta\mu_1} = 1 + \frac{1}{\beta} + A.$$

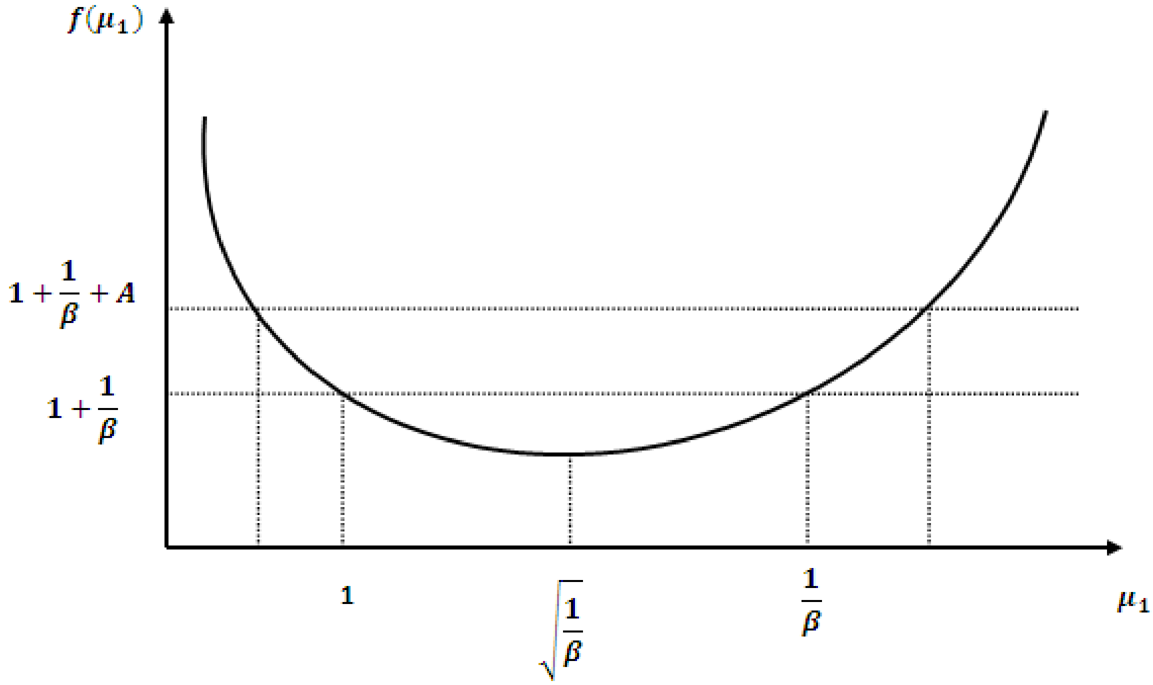
Then the argument goes as follows:

- The function $f(\mu_1) = \mu_1 + \frac{1}{\beta\mu_1}$ is continuous on the real line.
- It takes the same value at 1 and $\frac{1}{\beta}$:

$$f(1) = 1 + \frac{1}{\beta} = f(\frac{1}{\beta}).$$

- It has a global minimum at $\mu_1 = \sqrt{\frac{1}{\beta}}$, where $1 < \sqrt{\frac{1}{\beta}} < \frac{1}{\beta}$.
- Since $1+\frac{1}{\beta}+A > 1+\frac{1}{\beta}$ it follows that the two possible values of μ_1 satisfying $f(\mu_1) = 1+\frac{1}{\beta}+A$ are one below $\mu_1 = 1$ and the other above $\mu_1 = \frac{1}{\beta}$ (if you don't see why, just look at the graph below!).

■



The function $f(\mu_1)$ in the proof

The matrix D can be decomposed into $D = \Gamma\Lambda\Gamma^{-1}$, where Λ is a diagonal matrix with the eigenvalues of D on its diagonal, and where Γ is a matrix of the right eigenvectors. The condition of Blanchard-Kahn for finding a solution to this problem is that the number of eigenvalues that are outside the unit circle (which have an absolute value greater than one) is equal to the number of expectational variables. In our simple model such condition is satisfied (why?).

Premultiplying (15) by $\Gamma^{-1} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$, we get:

$$\Gamma^{-1}E_t s^0_{t+1} = \Lambda\Gamma^{-1}s^0_t + \Gamma^{-1}F\widehat{z}_t. \tag{16}$$

It is now useful to define the following transformed variables: $s^1_t = \Gamma^{-1}s^0_t$, and $Q = \Gamma^{-1}F$, hence

$$E_t s^1_{t+1} = \Lambda s^1_t + Qz_t \tag{17}$$

$$\Longrightarrow E_t \begin{bmatrix} s^1_{1,t+1} \\ s^1_{2,t+1} \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} s^1_{1,t} \\ s^1_{2,t} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \widehat{z}_t \tag{18}$$

where $s^1_{1,t} = u_1\widehat{k}_t + v_1\widehat{c}_t$ and $s^1_{2,t} = u_2\widehat{k}_t + v_2\widehat{c}_t$.

It is clear that to get a stable solution we need to solve the first equation backwards ($|\mu_1| < 1$) and the second equation forwards ($|\mu_2| > 1$). Let's start from the second one. Solving for $s^1_{2,t}$ we obtain:

$$s^1_{2,t} = \frac{1}{\mu_2} E_t s^1_{2,t+1} - \frac{Q_2}{\mu_2} \widehat{z}_t, \tag{19}$$

where $-1 < \frac{1}{\mu_2} < 1$. Iterating (19) forward and applying the law of iterated expectations, together with the fact that $E_t \widehat{z}_{t+j} = \rho^j \widehat{z}_t$, we get:

$$\begin{aligned} s^1_{2,t} &= \lim_{j \longrightarrow \infty} \left(\frac{1}{\mu_2}\right)^j E_t s^1_{2,t+j} - \frac{Q_2}{\mu_2} \sum_{j=0}^{\infty} \left(\frac{\rho}{\mu_2}\right)^j \widehat{z}_t = && (20) \\ &= \frac{Q_2}{\rho - \mu_2} \widehat{z}_t, && (21) \end{aligned}$$

which provides us with the stability condition for the optimization problem of the social planner:

$$s^1_{2,t} = \frac{Q_2}{\rho - \mu_2} \widehat{z}_t. \tag{22}$$

The definition of $s^1_{2,t} = u_2 \widehat{k}_t + v_2 \widehat{c}_t$ can be used to write the stability condition above as:

$$\widehat{c}_t = -\frac{u_2}{v_2} \widehat{k}_t + \frac{Q_2/v_2}{\rho - \mu_2} \widehat{z}_t. \tag{23}$$

Now substitute (23) into the first equation of the system and solve for $\widehat{k}_{t+1}$ as a function of $\widehat{k}_t$ and $\widehat{z}_t$:

$$\widehat{k}_{t+1} = \mu_1 \widehat{k}_t + G \widehat{z}_t, \tag{24}$$

where

$$G = \frac{Q_1 - \frac{\rho - \mu_1}{\rho - \mu_2} \frac{Q_2 v_1}{v_2}}{u_1 - v_1 u_2 / v_2}. \tag{25}$$

Finally we can represent the solution to the log-linear system in *state-space form:*

$$\begin{bmatrix} \widehat{k}_{t+1} \\ \widehat{z}_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_1 & G \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{z}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}, \tag{26}$$

$$\widehat{c}_t = \begin{bmatrix} -\frac{u_2}{v_2} & \frac{Q_2/v_2}{\rho - \mu_2} \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \widehat{z}_t \end{bmatrix}. \tag{27}$$

For future reference, let's define the following matrices:

$$\begin{aligned} \Pi &= \begin{bmatrix} \mu_1 & G \\ 0 & \rho \end{bmatrix}, \quad W = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ U &= \begin{bmatrix} -\frac{u_2}{v_2} & \frac{Q_2/v_2}{\rho - \mu_2} \end{bmatrix}. \end{aligned}$$

Given $\widehat{k}_0$ and $\widehat{z}_0$, we can generate artificial time series from this model through the following steps:

- Generate a sample realization for the productivity innovation $\{\varepsilon_t\}_{t=0}^T$;
- Iterate on equation (26) to get time series for $\left\{\widehat{k}_{t+1}, \widehat{z}_{t+1}\right\}_{t=0}^T$;
- The consumption time series $\{\widehat{c}_t\}_{t=0}^T$ can then be obtained from equation (27).

6 A Numerical Example

Let's apply what we just learnt to a specific example. The parameter values of our simple economy are:

Parameter	**Value**	**Description**
A	1	tecnological parameter
β	0.95	time discount factor
σ	3	inverse of intertemporal elasticity
δ	0.1	depreciation rate of capital
α	0.35	output elasticity of capital stock
ρ	0.95	Autoregressive parameter for the technology shock
σ_ε	0.01	Standard deviation for innovation in technology

The state-space form matrices are:

$$\begin{aligned} \Pi &= \begin{bmatrix} 0.9210 & 0.1985 \\ 0 & 0.9500 \end{bmatrix} \\ W &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ U &= \begin{bmatrix} 0.3665 & 0.6616 \end{bmatrix} \end{aligned}$$

A computationally more efficient solution method, that reaches *exactly* the same state-space representation delivered by Blanchard-Kahn is Uhlig's undetermined coefficients method, which we describe in the next section.

7 Uhlig's Undetermined Coefficients Approach

This approach starts from the log-linear approximation of the conditions characterizing the solution (10 to 12 obtained above):

$$\alpha_1\widehat{c}_t + \alpha_2\widehat{k}_{t+1} + \alpha_3\widehat{k}_t + \alpha_4\widehat{z}_t = 0 \tag{28}$$

$$\beta_1\widehat{c}_t + \beta_2 E_t\widehat{c}_{t+1} + \beta_3\widehat{z}_t + \beta_4\widehat{k}_{t+1} = 0 \tag{29}$$

where $\alpha_1 = \frac{c}{k}, \alpha_2 = 1, \alpha_3 = -\frac{1}{\beta}, \alpha_4 = -\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right), \beta_1 = \sigma, \beta_2 = -\sigma, \beta_3 = [1-\beta(1-\delta)]\rho, \beta_4 = -[1-\beta(1-\delta)](1-\alpha)$. The fact that there are two state variables in this economy, z_t and k_t, suggests that the two decision variables, c_t and k_{t+1}, should be each period function of the two states:

$$\widehat{k}_{t+1} = \eta_{kk}\widehat{k}_t + \eta_{kz}\widehat{z}_t, \tag{30}$$

$$\widehat{c}_t = \eta_{ck}\widehat{k}_t + \eta_{cz}\widehat{z}_t, \tag{31}$$

and also, taking conditional expectations as of time t,

$$E_t\widehat{k}_{t+1} = \widehat{k}_{t+1} \tag{32}$$

$$\begin{aligned} E_t\widehat{c}_{t+1} &= \eta_{ck}\widehat{k}_{t+1} + \eta_{cz}E_t\widehat{z}_{t+1} = \\ &= \eta_{ck}\left(\eta_{kk}\widehat{k}_t + \eta_{kz}\widehat{z}_t\right) + \eta_{cz}\rho\widehat{z}_t = \\ &\quad \eta_{ck}\eta_{kk}\widehat{k}_t + \left(\eta_{ck}\eta_{kz} + \eta_{cz}\rho\right)\widehat{z}_t. \end{aligned} \tag{33}$$

Plugging (30)-(33) into the log-linear approximation (28)-(29) and rearranging gives us

$$\left(\alpha_1\eta_{ck} + \alpha_2\eta_{kk} + \alpha_3\right)\widehat{k}_t + \left(\alpha_1\eta_{cz} + \alpha_2\eta_{kz} + \alpha_4\right)\widehat{z}_t = 0,$$

$$\left(\beta_1\eta_{ck} + \beta_2\eta_{ck}\eta_{kk} + \beta_4\eta_{kk}\right)\widehat{k}_t + \left[\beta_1\eta_{cz} + \beta_2\left(\eta_{ck}\eta_{kz} + \rho\eta_{cz}\right) + \beta_3 + \beta_4\eta_{kz}\right]\widehat{z}_t = 0,$$

and for these equations to hold we need to impose the following restrictions:

$$\alpha_1\eta_{ck} + \alpha_2\eta_{kk} + \alpha_3 = 0, \tag{34}$$

$$\alpha_1\eta_{cz} + \alpha_2\eta_{kz} + \alpha_4 = 0, \tag{35}$$

$$\beta_1\eta_{ck} + \beta_2\eta_{ck}\eta_{kk} + \beta_4\eta_{kk} = 0, \tag{36}$$

$$\beta_1\eta_{cz} + \beta_2\left(\eta_{ck}\eta_{kz} + \rho\eta_{cz}\right) + \beta_3 + \beta_4\eta_{kz} = 0. \tag{37}$$

From (34) we have:

$$\eta_{ck} = -\frac{\alpha_2\eta_{kk} + \alpha_3}{\alpha_1}, \tag{38}$$

which taken to (36) yields:

$$\beta_2\alpha_2\eta_{kk}^2 + \left(\beta_1\alpha_2 + \beta_2\alpha_3 - \beta_4\alpha_1\right)\eta_{kk} + \beta_1\alpha_3 = 0,$$

a quadratic equation in η_{kk}. which must be solved to obtain the value of this parameter. Equation (38) will then give us the value of η_{ck}. These two, taken to (35) and(37) will provide us with the values of η_{cz} and η_{kz}. Only one of the roots of the quadratic equation in η_{kk} is less than one[3]. That is the *stable* root: the other root would clearly produce an explosive path for k_t and it is not used. This is the way *stability* is imposed in this solution approach. After tedious algebra, the solution for the coefficients with respect to the technology shock z_t are:

$$\begin{aligned}\eta_{kz} &= -\frac{(\beta_1+\rho\beta_2)\,\alpha_4-\alpha_1\beta_3}{\beta_1\alpha_2-\beta_2\,(\alpha_1\eta_{ck}-\alpha_2\rho)-\beta_4\alpha_1},\\ \eta_{cz} &= -\frac{\alpha_2\eta_{kz}+\alpha_4}{\alpha_1}.\end{aligned}$$

Once we have the four η-parameters, generating the time series for the stock of capital, consumption and output is straightforward. Finally, as we did in section 5, we can represent the solution to the log-linear system in *state-space form*:

$$\begin{bmatrix}\widehat{k}_{t+1}\\ \widehat{z}_{t+1}\end{bmatrix}=\begin{bmatrix}\eta_{kk} & \eta_{kz}\\ 0 & \rho\end{bmatrix}\begin{bmatrix}\widehat{k}_{t}\\ \widehat{z}_{t}\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\varepsilon_{t+1}, \tag{39}$$

$$\widehat{c}_t=\begin{bmatrix}\eta_{ck} & \eta_{cz}\end{bmatrix}\begin{bmatrix}\widehat{k}_{t}\\ \widehat{z}_{t}\end{bmatrix}. \tag{40}$$

Exercise 3 *Check that applying Uhlig's solution method to our economy, using the same parameter values of section (6), delivers exactly the same state-space form representation derived under Blanchard-Kahn. (Hint: simply run the matlab file handout_ class2.m after typing solution_ method='Uhlig'.)*

Matlab files[4]

`handout_class2.m`: solves model using Blanchard-Kahn or Uhlig's undetermined coeff and generates simulated time series.

`uhlig.m`: solves model using uhlig by hand.

`uhlig_matrici.m`: solves model using uhlig with matrix set-up.

8 Appendix: derivation of log-linear approximation using Uhlig's calculus

Consider equation (2) that we reproduce here for convenience:

$$c_t+k_{t+1}=(1-\delta)k_t+z_tk_t^{\alpha}$$

[3] You can prove this by yourself using an argument similar to that used in section (5).

[4] You can find them on my personal webpage.

In the main text (see Section 4) we explained how to take a Taylor series expansion of the equilibrium equations. There is however a simpler method, originally proposed by Harald Uhlig, to obtain the loglinear approximation that avoids explicitely computing the derivatives. Let $\widehat{x}_t = \log\left(\frac{x_t}{\overline{x}}\right) \simeq \frac{x_t - \overline{x}}{\overline{x}}$ as before. Notice that

$$x_t = \overline{x}e^{\widehat{x}_t} \simeq \overline{x}(1+\widehat{x}_t)$$

Following this approach, the log-linearized system can be obtained as follows:

1. Multiply out everything before log-linearizing
2. Replace each variable x_t with $\overline{x}e^{\widehat{x}_t}$
3. Use the approximation $\overline{x}e^{\widehat{x}_t} \simeq \overline{x}(1+\widehat{x}_t)$ and $\widehat{x}_t\widehat{y}_t \simeq 0$.

Applying this method to equation (2) we get:

$$\begin{aligned}
c_t + k_{t+1} &= (1-\delta)k_t + z_t k_t^{\alpha} \\
\overline{c}e^{\widehat{c}_t} + \overline{k}e^{\widehat{k}_{t+1}} &= (1-\delta)\overline{k}e^{\widehat{k}_t} + \overline{z}e^{\widehat{z}_t}(\overline{k}e^{\widehat{k}_t})^{\alpha} \\
\overline{c}e^{\widehat{c}_t} + \overline{k}e^{\widehat{k}_{t+1}} &= (1-\delta)\overline{k}e^{\widehat{k}_t} + \overline{z}\overline{k}^{\alpha}e^{\widehat{z}_t+\alpha\widehat{k}_t} \\
\overline{c}(1+\widehat{c}_t) + \overline{k}(1+\widehat{k}_{t+1}) &= (1-\delta)\overline{k}(1+\widehat{k}_t) + \overline{z}\overline{k}^{\alpha}(1+\widehat{z}_t+\alpha\widehat{k}_t) \\
\overline{c}\widehat{c}_t + \overline{k}\widehat{k}_{t+1} &= (1-\delta)\overline{k}\widehat{k}_t + \overline{z}\overline{k}^{\alpha}\widehat{z}_t + \alpha\overline{z}\overline{k}^{\alpha}\widehat{k}_t \\
\frac{\overline{c}}{\overline{k}}\widehat{c}_t + \widehat{k}_{t+1} &= (1-\delta)\widehat{k}_t + \overline{z}\overline{k}^{\alpha-1}\widehat{z}_t + \alpha\overline{z}\overline{k}^{\alpha-1}\widehat{k}_t \\
\frac{\overline{c}}{\overline{k}}\widehat{c}_t + \widehat{k}_{t+1} &= [(1-\delta) + \alpha\overline{z}\overline{k}^{\alpha-1}]\widehat{k}_t + \overline{z}\overline{k}^{\alpha-1}\widehat{z}_t \\
\frac{\overline{c}}{\overline{k}}\widehat{c}_t + \widehat{k}_{t+1} &= \frac{1}{\beta}\widehat{k}_t + \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\widehat{z}_t
\end{aligned}$$

and this is equal to equation (8) in the main text.

Consider now equation (3). Applying Uhlig's method we get:

$$\begin{aligned}
c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma}\left(\alpha z_{t+1}k_{t+1}^{\alpha-1} + 1 - \delta\right) \\
\overline{c}^{-\sigma}e^{-\sigma\widehat{c}_t} &= \beta\overline{c}^{-\sigma}e^{-\sigma\widehat{c}_{t+1}}\left(\alpha\overline{z}\overline{k}^{\alpha-1}e^{\widehat{z}_{t+1}+(\alpha-1)\widehat{k}_{t+1}} + 1 - \delta\right) \\
e^{-\sigma\widehat{c}_t} &= \beta\alpha\overline{z}\overline{k}^{\alpha-1}e^{-\sigma\widehat{c}_{t+1}}e^{\widehat{z}_{t+1}+(\alpha-1)\widehat{k}_{t+1}} + \beta(1-\delta)e^{-\sigma\widehat{c}_{t+1}} \\
1-\sigma\widehat{c}_t &= \beta\alpha\overline{z}\overline{k}^{\alpha-1}(1-\sigma\widehat{c}_{t+1} + \widehat{z}_{t+1} + (\alpha-1)\widehat{k}_{t+1}) + \beta(1-\delta)(1-\sigma\widehat{c}_{t+1}) \\
-\sigma\widehat{c}_t &= \beta\alpha\overline{z}\overline{k}^{\alpha-1}(-\sigma\widehat{c}_{t+1} + \widehat{z}_{t+1} + (\alpha-1)\widehat{k}_{t+1}) + \beta(1-\delta)(-\sigma\widehat{c}_{t+1}) \\
-\sigma\widehat{c}_t &= -\sigma\widehat{c}_{t+1}[\beta\alpha\overline{z}\overline{k}^{\alpha-1} + \beta(1-\delta)] + \beta\alpha\overline{z}\overline{k}^{\alpha-1}\widehat{z}_{t+1} + (\alpha-1)\beta\alpha\overline{z}\overline{k}^{\alpha-1}\widehat{k}_{t+1}
\end{aligned}$$

Using the steady state relationship $\beta\alpha\overline{z}\overline{k}^{\alpha-1} + \beta(1-\delta) = 1$ delivers:

$$-\sigma\widehat{c}_t = -\sigma\widehat{c}_{t+1} + [1-\beta(1-\delta)]\widehat{z}_{t+1} - (1-\alpha)[1-\beta(1-\delta)]\widehat{k}_{t+1}$$

which is again equal to equation (9).

References

1. Malley, Jim, Lecture Notes on the Theory, Calibration & Estimation of Dynamic Stochastic General Equilibrium Models, mimeo, University of Glasgow 2004.

2. Uhlig, Harald, A toolkit for analysing nonlinear dynamic stochastic models easily, in Ramon Marimon and Andrew Scott, eds., Computational Methods for the Study of Dynamic Economics, Oxford: Oxford University Press, 1999, pp. 30 61.