Technical Appendix to “Bayesian Estimation of an Open-Economy Model with Imperfect Pass-Through”

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Abstract

These notes describes the log-linearization of the model in Adolfson (2007, JIE) in some detail along with some other stuff.

1 Profit maximization under flexible prices

1.1 Some preliminaries

Start with some preliminaries;

The aggregate production function of the final goods firm (which translates the differentiated intermediate products into a composite final good) follows

$$Y_t = \left[\int_0^1 Y_{it}^{\frac{1}{\lambda_{f,t}}} di \right]^{\lambda_{f,t}}, \quad 1 \leq \lambda_{f,t} < \infty. \tag{1}$$

The corresponding aggregate price index is obtained by solving the problem

$$\begin{aligned} \min_{Y_{it}} \quad Z &= \int_0^1 P_{it} Y_{it} \; di, \\ \text{s.t.} \quad Y_t &= \left[\int_0^1 Y_{it}^{\frac{1}{\lambda_{f,t}}} di \right]^{\lambda_{f,t}} = 1 \end{aligned} \tag{2}$$

The first order condition of this minimization follows

$$P_{it} - \check{\phi} \left[\lambda_{f,t} \left((Y_t)^{\frac{1}{\lambda_{f,t}}} \right)^{\lambda_{f,t}-1} \frac{1}{\lambda_{f,t}} (Y_{it})^{\frac{1}{\lambda_{f,t}}-1} \right] = 0. \tag{3}$$

$$Y_{it} = \left(\frac{P_{it}}{\check{\phi}} \right)^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} Y_t. \tag{4}$$

Inserting this into the composite consumption index (equation (1)) yields:

$$\begin{aligned} Y_t &= \left[\int_0^1 \left(\left(\frac{P_{it}}{\check{\phi}} \right)^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} Y_t \right)^{\frac{1}{\lambda_{f,t}}} di \right]^{\lambda_{f,t}} \\ &= \left(\frac{1}{\check{\phi}} \right)^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} \int_0^1 \left[(P_{it})^{\frac{1}{1-\lambda_{f,t}}} di \right]^{\lambda_{f,t}} Y_t. \end{aligned}$$

Simplifying:

$$\check{\phi}^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} = \left[\int_0^1 (P_{it})^{\frac{1}{1-\lambda_{f,t}}} \, di \right]^{\lambda_{f,t}}. \tag{5}$$

The Lagrange multiplier of the minimization problem in equation (2), $\check{\phi}$, can be defined as the aggregate price index since it is the cost of one extra unit of the composite good. Solving for ϕ yields

$$\check{\phi} = \left[\int_0^1 (P_{it})^{\frac{1}{1-\lambda_{f,t}}} \, di \right]^{1-\lambda_{f,t}} \equiv P_t. \tag{6}$$

The demand for each intermediate good is obtained by maximizing:

$$\begin{aligned} \max_{Y_{it}} \quad Y_t &= \left[\int_0^1 Y_{it}^{\frac{1}{\lambda_{f,t}}} di \right]^{\lambda_{f,t}}, \\ \text{s.t.} \quad Z &= \int_0^1 P_{it} Y_{it} \; di. \end{aligned} \tag{7}$$

The first order condition with respect to good i and j yields:

$$\lambda_{f,t} Y_t^{\lambda_{f,t}-1} \frac{1}{\lambda_{f,t}} Y_{it}^{\frac{1}{\lambda_{f,t}}-1} - \mu P_{it} = 0, \tag{8}$$

$$\lambda_{f,t} Y_t^{\lambda_{f,t}-1} \frac{1}{\lambda_{f,t}} Y_{jt}^{\frac{1}{\lambda_{f,t}}-1} - \mu P_{jt} = 0. \tag{9}$$

or,

$$\frac{Y_t^{\lambda_{f,t}-1} Y_{it}^{\frac{1-\lambda_{f,t}}{\lambda_{f,t}}}}{P_{it}} = \frac{Y_t^{\lambda_{f,t}-1} Y_{jt}^{\frac{1-\lambda_{f,t}}{\lambda_{f,t}}}}{P_{jt}}. \tag{10}$$

The relative demand for the two products can then be written:

$$Y_{it} = \left(\frac{P_{jt}}{P_{it}}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{jt}. \tag{11}$$

Inserting this into the expenditure function above yields:

$$\begin{aligned} Z &= \int_0^1 P_{it} \left(\frac{P_{jt}}{P_{it}}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{jt} \quad di & (12) \\ &= (P_{jt})^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{jt} \left[\int_0^1 (P_{it})^{\frac{1}{1-\lambda_{f,t}}} di\right]. & (13) \end{aligned}$$

Use the aggregate price index in equation (6) to rewrite this as

$$Z = (P_{jt})^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{jt} (P_t)^{\frac{1}{1-\lambda_{f,t}}}. \tag{14}$$

Inserting this into equation (11) implies:

$$\begin{aligned} Y_{it} &= (P_{it})^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} Z (P_t)^{\frac{-1}{1-\lambda_{f,t}}}. \\ &= (P_{it})^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} \frac{Z}{P_t} P_t (P_t)^{\frac{-1}{1-\lambda_{f,t}}}. \\ &= \left(\frac{P_{it}}{P_t}\right)^{\frac{\lambda_{f,t}}{1-\lambda_{f,t}}} \frac{Z}{P_t}. & (15) \end{aligned}$$

Also using that $Z = P_t Y_t$ implies that the demand for firm i's product can be written as

$$Y_{it} = \left(\frac{P_t}{P_{it}}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_t. \tag{16}$$

We also want to define the price elasticity of demand which is useful in the profit maximization. Differentiate equation (16) with respect to the price

$$\frac{\partial Y_{it}}{\partial P_{it}} = \frac{\lambda_{f,t}}{1-\lambda_{f,t}} \left(P_{it}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}-1} \left(P_t\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_t.$$

Multiply this expression by P_{it} and divide by Y_{it} to obtain the price elasticity of demand

$$\begin{aligned} \frac{\partial Y_{it}}{\partial P_{it}} \frac{P_{it}}{Y_{it}} &= \frac{\lambda_{f,t}}{1-\lambda_{f,t}} \frac{P_{it}}{P_{it}} \frac{1}{Y_{it}} \left(\frac{P_t}{P_{it}}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_t \\ &= \frac{\lambda_{f,t}}{1-\lambda_{f,t}}. \end{aligned} \tag{17}$$

1.2 The domestic firms' optimization problem

Under *flexible prices* the domestic firm i faces the following maximization problem every period

$$\max_{P_{it}} \qquad P_{it} Y_{it} - MC_t Y_{it} - MC_t \phi. \tag{18}$$

The fixed cost ϕ (in real terms) enters the production function in order to make profits zero in steady state. However, we do not want to make it affect the pricing decision in this maximization problem why it is translated into nominal terms by MC_t. [Can it be entirely redundant in the flex price case?]

The first order condition yields

$$Y_{it} + P_{it} \frac{\partial Y_{it}}{\partial P_{it}} - MC_t \frac{\partial Y_{it}}{\partial P_{it}} = 0. \tag{19}$$

Multiply by P_{it} and divide by Y_{it} to obtain

$$P_{it} + P_{it}\frac{\partial Y_{it}}{\partial P_{it}}\frac{P_{it}}{Y_{it}} - MC_t\frac{\partial Y_{it}}{\partial P_{it}}\frac{P_{it}}{Y_{it}} = 0, \tag{20}$$

$$P_{it}(1 + \frac{\partial Y_{it}}{\partial P_{it}}\frac{P_{it}}{Y_{it}}) = MC_t\frac{\partial Y_{it}}{\partial P_{it}}\frac{P_{it}}{Y_{it}}. \tag{21}$$

Insert equation (17), which yields

$$P_{it}(1 + \frac{\lambda_{f,t}}{1-\lambda_{f,t}}) = MC_t\frac{\lambda_{f,t}}{1-\lambda_{f,t}}. \tag{22}$$

Simplify to obtain the standard first order condition of a monopolist

$$P_{it} = \lambda_{f,t} MC_t, \tag{23}$$

where the price is set as a markup ($\lambda_{f,t}$) over marginal cost.

1.2.1 Cost minimization

Assume now that $P_{i,t}$ is given. Then the firm must produce $Y_{i,t}$. The domestic firm i's cost minimization in period t follows:

$$\min_{K_{i,t},X_{i,t}} W_t R_t^f X_{i,t} + R_t^k K_{i,t} + \lambda_t P_{i,t}\left[Y_{i,t} - z_t^{1-\alpha}\epsilon_t K_{i,t}^{\alpha} X_{i,t}^{1-\alpha} + z_t\phi\right]. \tag{24}$$

The first order conditions with respect to $X_{i,t}$ and $K_{i,t}$ are

$$W_t R_t^f = (1-\alpha)\lambda_t P_{i,t} z_t^{1-\alpha}\epsilon_t K_{i,t}^{\alpha} X_{i,t}^{-\alpha} \tag{25}$$

$$R_t^k = \alpha\lambda_t P_{i,t} z_t^{1-\alpha}\epsilon_t K_{i,t}^{\alpha-1} X_{i,t}^{1-\alpha} \tag{26}$$

Combining these two equations yield the following condition

$$\frac{X_{i,t}}{K_{i,t}} = \frac{R_t^k}{W_t R_t^f}\frac{1-\alpha}{\alpha}. \tag{27}$$

Solve for $\lambda_{i,t}$ in equation (26):

$$\lambda_t = \frac{1}{\alpha} R_t^k \frac{1}{P_{i,t} z_t^{1-\alpha} \epsilon_t} \left(\frac{X_{i,t}}{K_{i,t}} \right)^{\alpha - 1}. \tag{28}$$

Use the conditon above to obtain:

$$\begin{aligned} \lambda_t &= \frac{1}{\alpha} R_t^k \frac{1}{P_{i,t} z_t^{1-\alpha} \epsilon_t} \left(\frac{R_t^k}{W_t R_t^f} \frac{1-\alpha}{\alpha} \right)^{\alpha-1} & (29) \\ &= \left(\frac{1}{1-\alpha} \right)^{1-\alpha} \left(\frac{1}{\alpha} \right)^{\alpha} \left(R_t^k \right)^{\alpha} \left(W_t R_t^f \right)^{1-\alpha} \frac{1}{z_t^{1-\alpha} \epsilon_t} \frac{1}{P_{i,t}}. & (30) \end{aligned}$$

$\lambda_t P_{i,t}$can be interpreted as the nominal marginal cost since this is the nominal cost of producing one extra unit of the domestic good. Consequently, λ_t is the real marginal cost.

1.3 The importing firms' optimization problem

Equivalently under *flexible prices*, an importing firm i faces the following maximization problem:

$$\max_{P_{it}^{m,c}} \qquad P_{it}^{m,c} C_{it}^m - S_t P_t^* C_{it}^m. \tag{31}$$

The first order condition for the price of consumption goods yields after some rearranging

$$P_{it}^{m,c} (1 + \frac{\partial C_{it}^m}{\partial P_{it}^{m,c}} \frac{P_{it}^{m,c}}{C_{it}^m}) = S_t P_t^* \frac{\partial C_{it}^m}{\partial P_{it}^{m,c}} \frac{P_{it}^{m,c}}{C_{it}^m}. \tag{32}$$

The demand for firm i's product follows

$$\begin{aligned} C_{it}^m &= \left(\frac{P_{it}^{m,c}}{P_t^{m,c}} \right)^{-\eta_t^{m,c}} C_t^m \\ &= \left(\frac{P_{it}^{m,c}}{P_t^{m,c}} \right)^{-\eta_t^{m,c}} \omega_c \left(\frac{P_t^m}{P_t^c} \right)^{-\eta_c} C_t. \end{aligned}$$

Differentiate this with respect to $P_{it}^{m,c}$, and rearrange to obtain the price elasticity of demand firm i faces:

$$\begin{aligned} \frac{\partial C_{it}^m}{\partial P_{it}^{m,c}} \frac{P_{it}^{m,c}}{C_{it}^m} &= -\eta_t^{m,c} \left(\frac{P_{it}^{m,c}}{P_t^{m,c}} \right)^{-\eta_t^{m,c}-1} \frac{1}{P_t^{m,c}} C_t^m \frac{P_{it}^{m,c}}{C_{it}^m} \\ &= -\eta_t^{m,c}. \end{aligned}$$

Insert this in the first order condition above, which yields the standard monopolistic price setting:

$$P_{it}^{m,c} = \frac{\eta_t^{m,c}}{\eta_t^{m,c} - 1} S_t P_t^*. \tag{33}$$

2 Profit maximization under sticky prices

2.1 Import price determination using Calvo price setting

The composite import products that enters the households' consumption and investment are defined as follows

$$C_t^m = \left[\int_0^1 \left(C_{it}^m \right)^{\frac{\eta_t^{m,c}-1}{\eta_t^{m,c}}} di \right]^{\frac{\eta_t^{m,c}}{\eta_t^{m,c}-1}}, \quad \eta_t^{m,c} > 1, \tag{34}$$

$$I_t^m = \left[\int_0^1 \left(I_{it}^m \right)^{\frac{\eta_t^{m,i}-1}{\eta_t^{m,i}}} di \right]^{\frac{\eta_t^{m,i}}{\eta_t^{m,i}-1}}, \quad \eta_t^{m,i} > 1, \tag{35}$$

with the corresponding price indices:

$$P_t^{m,c} = \left[\int_0^1 \left(P_{it}^{m,c} \right)^{1-\eta_t^{m,c}} di \right]^{\frac{1}{1-\eta_t^{m,c}}}, \tag{36}$$

$$P_t^{m,i} = \left[\int_0^1 \left(P_{it}^{m,i} \right)^{1-\eta_t^{m,i}} di \right]^{\frac{1}{1-\eta_t^{m,i}}}, \tag{37}$$

There exists a continuum of importing intermediate (consumption and investment) firms that supplies these differentiated goods. Each of them faces a random probability $(1-\xi^m)$ that he/she can change (i.e. reoptimize) her price in any period. Since all firms are alike the subscript i can be dropped. Let the reoptimized price for an imported consumption good (investment good) be denoted $P_{new,t}^{m,c}(P_{new,t}^{m,i})$. With probability ξ^m the firm does not reoptimize, and its price in period $t+1$ follows $P_{t+1}^{m,c} = \pi_t^{m,c} P_{new,t}^{m,c}$ $(P_{t+1}^{m,c} = \pi_t^{m,i} P_{new,t}^{m,i})$. The price in period $t+j$ is then $\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c}$. The consumption and investment firm face the following optimization problem, respectively:

$$\max_{P_{new,t}^{m,c}} \mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j}[\left(\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c}\right) C_{it+j}^m \\ -S_{t+j}P_{t+j}^* C_{it+j}^m - S_{t+j}P_{t+j}^* z_{t+j}\phi^{m,c}], \tag{38}$$

$$\max_{P_{new,t}^{m,i}} \mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j}[\left(\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,i}\right) I_{it+j}^m \\ -S_{t+j}P_{t+j}^* I_{it+j}^m - S_{t+j}P_{t+j}^* z_{t+j}\phi^{m,i}], \tag{39}$$

where β is a stochastic discount factor. Profits are also discounted by υ_t, the marginal utility of the nominal income because profits are maximized conditional upon the utility of the households since households and firms pool their resources in the end of each period?? [can I show which is the relevant discount factor?! - does it depend on taxes?; see eq. (3.5 in Jesper's notes)]. [If we allow a positive markup we need to deduct a fixed cost here, or we transfer the profits into the household budget constraint!!]. $\phi^{m,c}$ and $\phi^{m,i}$ are fixed costs (in real terms) that enter the firms' optimization problem because we want to impose that import profits are zero in steady state.

The demand for an imported consumption good i and an investment good i, respectively is given by

$$
\begin{aligned}
C_{it}^{m} &= \left(\frac{P_{it}^{m,c}}{P_{t}^{m,c}}\right)^{-\eta_t^{m,c}} C_t^m, \\
I_{it}^{m} &= \left(\frac{P_{it}^{m,i}}{P_{t}^{m,i}}\right)^{-\eta_t^{m,i}} I_t^m,
\end{aligned}
$$

Inserting this in the maximization problem above, and rearranging

$$
\begin{aligned}
&\max_{P_{new,t}^{c,m}} \mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j}[\left(\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c}\right)^{1-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m \\
&\qquad -S_{t+j}P_{t+j}^{*}[\left(\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m]]. \qquad (40) \\
&\max_{P_{new,t}^{i,m}} \mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j}[\left(\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,i}\right)^{1-\eta_{t+j}^{m,i}} \left(P_{t+j}^{m,i}\right)^{\eta_{t+j}^{m,i}} I_{t+j}^m \\
&\qquad -S_{t+j}P_{t+j}^{*}[\left(\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,i}\right)^{-\eta_{t+j}^{m,i}} \left(P_{t+j}^{m,i}\right)^{\eta_{t+j}^{m,i}} I_{t+j}^m]]. \qquad (41)
\end{aligned}
$$

The first order condition with respect to the price of the consumption good in equation (40) is

$$
\begin{aligned}
&\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j}[(1-\eta_{t+j}^{m,c}) \left(P_{new,t}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)^{1-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m \\
&\qquad +\eta_{t+j}^{m,c} \left(P_{new,t}^{m,c}\right)^{-\eta_{t+j}^{m,c}-1} \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m S_{t+j}P_{t+j}^{*}] = 0. \qquad (42) \\
&\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j} \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m \left(P_{new,t}^{m,c}\right)^{-\eta_{t+j}^{m,c}-1} \times \\
&\qquad [(1-\eta_{t+j}^{m,c}) \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c} + \eta_{t+j}^{m,c} S_{t+j}P_{t+j}^{*}] = 0.
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j} \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m \left(P_{new,t}^{m,c}\right)^{-\eta_{t+j}^{m,c}-1} \times \\
&\qquad [\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c} - \frac{\eta_{t+j}^{m,c}}{(\eta_{t+j}^{m,c}-1)} S_{t+j}P_{t+j}^{*}] = 0.
\end{aligned}
$$

Multiply through by $\left(P_{new,t}^{m,c}\right)^{-(-\eta_{t+j}^{m,c}-1)}$

$$\begin{aligned} &\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j} \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m \times \\ &[\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c} - \frac{\eta_{t+j}^{m,c}}{(\eta_{t+j}^{m,c}-1)} S_{t+j} P_{t+j}^*] = 0. \end{aligned} \tag{43}$$

To rewrite this in terms of stationary variables, multiply and divide by $P_{t+j}^{m,c}$:

$$\begin{aligned} &\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j} \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(P_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m P_{t+j}^{m,c} \times \\ &[\frac{\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c}}{P_{t+j}^{m,c}} - \frac{\eta_{t+j}^{m,c}}{(\eta_{t+j}^{m,c}-1)} \frac{S_{t+j} P_{t+j}^*}{P_{t+j}^{m,c}}] = 0. \end{aligned} \tag{44}$$

Rearrange and use the fact that $P_{t+j}^{m,c} \equiv P_t^{m,c}\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c}$:

$$\begin{aligned} &\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j} P_{t+j}^{m,c} \left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)^{-\eta_{t+j}^{m,c}} \left(P_t^{m,c}\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c}\right)^{\eta_{t+j}^{m,c}} C_{t+j}^m \times \\ &[\frac{\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right) P_{new,t}^{m,c}}{P_t^{m,c}\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c}} - \frac{\eta_{t+j}^{m,c}}{(\eta_{t+j}^{m,c}-1)} \frac{S_{t+j} P_{t+j}^*}{P_{t+j}^{m,c}}] = 0. \end{aligned} \tag{45}$$

Multiply through by $(P_t^{m,c})^{-\eta_{t+j}^{m,c}}$. This implies

$$\begin{aligned} &\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \psi_{t+j}^m \left(\frac{\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)}{(\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c})}\right)^{-\eta_{t+j}^{m,c}} C_{t+j}^m \times \\ &[\frac{\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)}{(\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c})} \frac{P_{new,t}^{m,c}}{P_t^{m,c}} - \frac{\eta_{t+j}^{m,c}}{(\eta_{t+j}^{m,c}-1)} \frac{S_{t+j} P_{t+j}^*}{P_{t+j}^{m,c}}] = 0, \end{aligned} \tag{46}$$

where $\psi_{t+j}^m \equiv \upsilon_{t+j} P_{t+j}^{m,c}$. Stationarize with the technology shock z_t:

$$\begin{aligned} &\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \psi_{t+j}^m \left(\frac{\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)}{(\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c})}\right)^{-\eta_{t+j}^{m,c}} z_{t+j} C_{t+j}^{\#} \times \\ &[\frac{\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)}{(\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c})} \frac{P_{new,t}^{m,c}}{P_t^{m,c}} - \frac{\eta_{t+j}^{m,c}}{(\eta_{t+j}^{m,c}-1)} \frac{S_{t+j} P_{t+j}^*}{P_{t+j}^{m,c}}] = 0, \end{aligned} \tag{47}$$

where $C_{t+j}^{\#} = \frac{C_{t+j}^m}{z_{t+j}}$.

Equivalently, the first order condition with respect to the imported investment good is:

$$\begin{aligned} &\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \upsilon_{t+j} \left(\pi_t^{m,i}\pi_{t+1}^{m,i}...\pi_{t+j-1}^{m,i}\right)^{-\eta_{t+j}^{m,i}} \left(P_{t+j}^{m,i}\right)^{\eta_{t+j}^{m,i}} I_{t+j}^m \times \\ &[\left(\pi_t^{m,i}\pi_{t+1}^{m,i}...\pi_{t+j-1}^{m,i}\right) P_{new,t}^{m,i} - \frac{\eta_{t+j}^{m,i}}{(\eta_{t+j}^{m,i}-1)} S_{t+j} P_{t+j}^*] = 0. \end{aligned} \tag{48}$$

or, in terms of stationary variables:

$$\begin{aligned} &\mathrm{E}_t \sum_{j=0}^{\infty} (\beta\xi^m)^j \psi_{t+j}^i \left(\frac{\left(\pi_t^{m,i}\pi_{t+1}^{m,i}...\pi_{t+j-1}^{m,i}\right)}{(\pi_{t+1}^{m,i}\pi_{t+2}^{m,i}...\pi_{t+j}^{m,i})}\right)^{-\eta_{t+j}^{m,i}} z_{t+j} I_{t+j}^{\#} \times \\ &[\frac{\left(\pi_t^{m,i}\pi_{t+1}^{m,i}...\pi_{t+j-1}^{m,i}\right)}{(\pi_{t+1}^{m,i}\pi_{t+2}^{m,i}...\pi_{t+j}^{m,i})} \frac{P_{new,t}^{m,i}}{P_t^{m,i}} - \frac{\eta_{t+j}^{m,i}}{(\eta_{t+j}^{m,i}-1)} \frac{S_{t+j} P_{t+j}^*}{P_{t+j}^{m,i}}] = 0, \end{aligned} \tag{49}$$

where $\psi^i_{t+j} = \upsilon_{t+j} P^i_{t+j}$.

2.1.1 Log-linearizing the import price equation (consumption good)

Rewriting the import price equation for the consumption good:

$$\begin{aligned}&\mathrm{E}_t \sum_{j=0}^{\infty} (\beta \xi^m)^j \, \psi^m_{z,t+j} \left(\frac{\left(\pi^{m,c}_t \pi^{m,c}_{t+1} \ldots \pi^{m,c}_{t+j-1}\right)}{\left(\pi^{m,c}_{t+1} \pi^{m,c}_{t+2} \ldots \pi^{m,c}_{t+j}\right)} \right)^{-\eta^{m,c}_{t+j}} C^{\#}_{t+j} \times \\ &\left[\frac{\left(\pi^{m,c}_t \pi^{m,c}_{t+1} \ldots \pi^{m,c}_{t+j-1}\right)}{\left(\pi^{m,c}_{t+1} \pi^{m,c}_{t+2} \ldots \pi^{m,c}_{t+j}\right)} \widetilde{p}^{m,c}_t - \frac{\eta^{m,c}_{t+j}}{(\eta^{m,c}_{t+j} - 1)} mc^{m,c}_{t+j} \right] = 0, \end{aligned} \tag{50}$$

where $\psi^m_{z,t+j} = \upsilon_{t+j} P^{m,c}_{t+j} z_{t+j}$, $\widetilde{p}^{m,c}_t = \frac{P^{m,c}_{new,t}}{P^{m,c}_t}$, and $mc^{m,c}_{t+j} = \frac{S_{t+j} P^*_{t+j}}{P^{m,i}_{t+j}}$.We will use a first order Taylor approximation around the steady state to log-linearize the price equation. This is obtained by totally differentiating equation (47) and evaluating it in steady-state:[1]

$$dF = F_{\tilde{p}} d\widetilde{p}^{m,c}_t + F_{mc} dmc^{m,c}_{t+j} + F_{\pi^m} d\pi^{m,c}_t + F_{\pi^m} d\pi^{m,c}_{t+j} + F_{\psi} d\psi^m_{z,t+j} + F_C dC^{\#}_{t+j} + F_{\eta} d\eta^{m,c}_{t+j}. \tag{51}$$

$$\begin{aligned}\frac{dF}{F} &= F_{\tilde{p}} d\widetilde{p}^{m,c}_t \frac{\widetilde{p}^{m,c}}{\widetilde{p}^{m,c} F} + F_{mc} dmc^{m,c}_{t+j} \frac{mc^{m,c}}{mc^{m,c} F} + F_{\pi^m} d\pi^{m,c}_t \frac{\pi^{m,c}}{\pi^{m,c} F} \\ &+ F_{\pi^m} d\pi^{m,c}_{t+j} \frac{\pi^{m,c}}{\pi^{m,c} F} + F_{\psi} d\psi^m_{z,t+j} \frac{\psi^m_z}{\psi^m_z F} + F_C dC^{\#}_{t+j} \frac{C^{\#}}{C^{\#} F} + F_{\eta} d\eta^{m,c}_{t+j} \frac{\eta^{m,c}}{\eta^{m,c} F}\end{aligned} \tag{52}$$

$$\begin{aligned}0 &= \widetilde{p}^{m,c} F_{\tilde{p}} \frac{d\widetilde{p}^{m,c}_t}{\widetilde{p}^{m,c}} + mc^{m,c} F_{mc} \frac{dmc^{m,c}_{t+j}}{mc^{m,c}} + \pi^{m,c} F_{\pi^m} \frac{d\pi^{m,c}_t}{\pi^{m,c}} + \pi^{m,c} F_{\pi^m} \frac{d\pi^{m,c}_{t+j}}{\pi^{m,c}} \\ &+ \psi^m_z F_{\psi} \frac{d\psi^m_{z,t+j}}{\psi^m_z} + C^{\#} F_C \frac{dC^{\#}_{t+j}}{C^{\#}} + \eta^{m,c} F_{\eta} \frac{d\eta^{m,c}_{t+j}}{\eta^{m,c}}.\end{aligned} \tag{53}$$

[1] To see this, note the following. A first order Taylor approximation of $G(X_t, Y_t) = \ln F(X_t, Y_t) = \ln F(e^{\ln X_t}, e^{\ln Y_t})$ around the steady state implies:

$G \simeq G(e^{\ln X}, e^{\ln Y}) + G_X (\ln X_t - \ln X) + G_Y (\ln Y_t - \ln Y)$,

$\ln\left(\frac{F(X_t, Y_t)}{F(X,Y)}\right) \simeq \frac{1}{F(X,Y)} F_x X (\ln X_t - \ln X) + \frac{1}{F(X,Y)} F_Y Y (\ln Y_t - \ln Y)$.

This is equivalent to: $\frac{dF(X_t, Y_t)}{F(X,Y)} \simeq \frac{1}{F(X,Y)} F_x X \frac{dX_t}{X} + \frac{1}{F(X,Y)} F_Y Y \frac{dY_t}{Y}$.

$$0=\widetilde{p}^{m,c}F_{\widetilde{p}}\widehat{\widetilde{p}}_t^{m,c}+mc^{m,c}F_{mc}\widehat{mc}_{t+j}^{m,c}+\pi^{m,c}F_{\pi^m}\widehat{\pi}_t^{m,c}+\pi^{m,c}F_{\pi^m}\widehat{\pi}_{t+j}^{m,c}$$

$$+\psi_z^m F_\psi\widehat{\psi}_{z,t+j}^m+C^{\#}F_C\widehat{C}_{t+j}^{\#}+\eta^{m,c}F_\eta d\widehat{\eta}_{t+j}^{m,c}, \qquad (54)$$

where a hat denotes the log deviation from steady state (i.e., $x_t = \frac{dx_t}{x} = \ln X_t - \ln X$). Note that in steady-state; $C_{t+j}^{\#}$ and $\psi_{z,t+j}^m$ are constants, $C^{\#}$ and ψ_z^m,respectively, and $\widetilde{p}^{m,c} = \frac{\eta^{m,c}}{(\eta^{m,c}-1)}mc^{m,c}, \pi_t^{m,c} = \overline{\pi}$, and $\frac{\left(\pi_t^{m,c}\pi_{t+1}^{m,c}...\pi_{t+j-1}^{m,c}\right)}{\left(\pi_{t+1}^{m,c}\pi_{t+2}^{m,c}...\pi_{t+j}^{m,c}\right)} = \frac{\pi_t^{m,c}}{\pi_{t+j}^{m,c}} = \frac{\overline{\pi}}{\overline{\pi}} = 1$.

Start by differentiating equation (47) with respect to the variables $\widetilde{p}_t^{m,c}$, $mc_{t+j}^{m,c}$, $\pi_t^{m,c}$, $\pi_{t+j}^{m,c}$, $\psi_{z,t+j}^m$, $C_{t+j}^{\#}$, and $\eta_{t+j}^{m,c}$, and evaluating all the first order conditions in steady-state:

$$\begin{aligned}\frac{\partial F}{\partial \widetilde{p}_t^{m,c}} &= \sum_{j=0}^{\infty}\left(\beta\xi^m\right)^j\psi_z^m C^{\#} \\ &= \frac{1}{(1-\beta\xi^m)}\psi_z^m C^{\#}.\end{aligned} \qquad (55)$$

$$\frac{\partial F}{\partial mc_{t+j}^{m,c}} = -\left(\beta\xi^m\right)^j\psi_z^m C^{\#}\frac{\eta^{m,c}}{(\eta^{m,c}-1)}, \qquad \forall \quad j\geq 0,$$

i.e.,

$$\frac{\partial F}{\partial mc_{t+j}^{m,c}} = -\sum_{j=0}^{\infty}\left(\beta\xi^m\right)^j\psi_z^m C^{\#}\frac{\eta^{m,c}}{(\eta^{m,c}-1)}. \qquad (56)$$

Regarding the derivatives with respect to $\pi_t^{m,c}$ and $\pi_{t+j}^{m,c}$, note that $\widetilde{p}^{m,c} = \frac{\eta^{m,c}}{(\eta^{m,c}-1)}mc^{m,c}$ in steady state, why we only have to differentiate with respect to the $\pi^{m,c}-$terms within brackets. Note also that $\widetilde{p}^{m,c}=1$ in steady state.

$$\begin{aligned}\frac{\partial F}{\partial \pi_t^{m,c}} &= \sum_{j=1}^{\infty}\left(\beta\xi^m\right)^j\psi_z^m C^{\#}\left(\frac{1}{\overline{\pi}}\right) \\ &= \frac{\beta\xi^m}{(1-\beta\xi^m)}\psi_z^m C^{\#}\left(\frac{1}{\overline{\pi}}\right).\end{aligned} \qquad (57)$$

$$\frac{\partial F}{\partial \pi_{t+j}^{m,c}} = \left(\beta\xi^m\right)^j\psi_z^m C^{\#}\left(\frac{-1}{\overline{\pi}}\right), \qquad \forall \quad j>0,$$

i.e.,

$$\frac{\partial F}{\partial \pi^{m,c}_{t+j}} = -\sum_{j=1}^{\infty} (\beta \xi^m)^j \, \psi^m_z C^{\#} \left(\frac{1}{\overline{\pi}} \right). \tag{58}$$

Note again that bracket is zero in steady state (i.e. $\widetilde{p}^{m,c} = \frac{\eta^{m,c}}{(\eta^{m,c}-1)} mc^{m,c}$) so the derivatives with respect to $\psi^m_{z,t+j}$ and $C^{\#}_{t+j}$ are zero;

$$\begin{aligned} \frac{\partial F}{\partial \psi^m_{z,t+j}} &= 0, \qquad (59) \\ \frac{\partial F}{\partial C^{\#}_{t+j}} &= 0. \qquad (60) \end{aligned}$$

Since the bracket is zero in steady state this implies that we only have to differentiate with respect to the $\eta^{m,c}_{t+j}-$terms within brackets.

$$\begin{aligned} \frac{\partial F}{\partial \eta^{m,c}_{t+j}} &= (\beta \xi^m)^j \, \psi^m_z C^{\#} mc^{m,c} \left[-1(\eta^{m,c}-1)^{-1} - \eta^{m,c}(-1)(\eta^{m,c}-1)^{-2} \right] \\ &= \frac{(\beta \xi^m)^j \, \psi^m_z C^{\#} mc^{m,c}}{(\eta^{m,c}-1)^2} \left[-(\eta^{m,c}-1) + \eta^{m,c} \right] \\ &= \frac{(\beta \xi^m)^j \, \psi^m_z C^{\#} mc^{m,c}}{(\eta^{m,c}-1)^2}. \end{aligned}$$

i.e.,

$$\frac{\partial F}{\partial \eta^{m,c}_{t+j}} = \sum_{j=0}^{\infty} \frac{(\beta \xi^m)^j \, \psi^m_z C^{\#} mc^{m,c}}{(\eta^{m,c}-1)^2}. \tag{61}$$

Collecting terms and inserting in equation (53):

$$\begin{aligned} 0 = {} & \frac{1}{(1-\beta\xi^m)} \psi^m_z C^{\#} \widetilde{p}^{m,c} \frac{d\widetilde{p}^{m,c}_t}{\widetilde{p}^{m,c}} - \sum_{j=0}^{\infty} (\beta \xi^m)^j \, \psi^m_z C^{\#} \frac{\eta^{m,c}}{(\eta^{m,c}-1)} mc^{m,c} \frac{dmc^{m,c}_{t+j}}{mc^{m,c}} \\ & + \frac{\beta \xi^m}{(1-\beta\xi^m)} \psi^m_z C^{\#} \left(\frac{1}{\overline{\pi}} \right) \overline{\pi} \frac{d\pi^{m,c}_t}{\overline{\pi}} - \sum_{j=1}^{\infty} (\beta \xi^m)^j \, \psi^m_z C^{\#} \left(\frac{1}{\overline{\pi}} \right) \overline{\pi} \frac{d\pi^{m,c}_{t+j}}{\overline{\pi}} \\ & + \sum_{j=0}^{\infty} \frac{(\beta \xi^m)^j \, \psi^m_z C^{\#} mc^{m,c}}{(\eta^{m,c}-1)^2} \eta^{m,c} \frac{d\eta^{m,c}_{t+j}}{\eta^{m,c}}. \end{aligned} \tag{62}$$

Rearranging, the Taylor expansion is;

$$\begin{aligned} 0 &= \frac{\psi_z^m C^{\#}}{(1-\beta\xi^m)}\widehat{\widetilde{p}}_t^{m,c} + \frac{\beta\xi^m\psi_z^m C^{\#}}{(1-\beta\xi^m)}\widehat{\pi}_t^{m,c} \\ &-\sum_{j=1}^{\infty}(\beta\xi^m)^j\,\psi_z^m C^{\#}\widehat{\pi}_{t+j}^{m,c} \\ &-\sum_{j=0}^{\infty}(\beta\xi^m)^j\,\psi_z^m C^{\#}\widetilde{p}^{m,c}\widehat{mc}_{t+j}^{m,c} \\ &+\sum_{j=0}^{\infty}\frac{(\beta\xi^m)^j\,\psi_z^m C^{\#}}{(\eta^{m,c}-1)^2}\widetilde{p}^{m,c}\widehat{\eta}_{t+j}^{m,c}. \end{aligned}$$

Solving for $\widehat{\widetilde{p}}_t$

$$\begin{aligned} \widehat{\widetilde{p}}_t &= \frac{(1-\beta\xi^m)}{\psi_z^m C^{\#}}[\frac{(-\beta\xi^m)\psi_z^m C^{\#}}{(1-\beta\xi^m)}\widehat{\pi}_t^{m,c} + \sum_{j=1}^{\infty}(\beta\xi^m)^j\,\psi_z^m C^{\#}\widehat{\pi}_{t+j}^{m,c} \\ &+\sum_{j=0}^{\infty}(\beta\xi^m)^j\,\psi_z^m C^{\#}\widehat{mc}_{t+j}^{m,c} - \sum_{j=0}^{\infty}\frac{(\beta\xi^m)^j\,\psi_z^m C^{\#}}{(\eta^{m,c}-1)^2}\widehat{\eta}_{t+j}^{m,c}]. \end{aligned} \tag{63}$$

$$\widehat{\widetilde{p}}_t = -\beta\xi^m\widehat{\pi}_t^{m,c} + (1-\beta\xi^m)\sum_{j=1}^{\infty}(\beta\xi^m)^j\,\widehat{\pi}_{t+j}^{m,c} + (1-\beta\xi^m)\sum_{j=0}^{\infty}(\beta\xi^m)^j\left(\widehat{mc}_{t+j}^{m,c} - \frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_{t+j}^{m,c}\right). \tag{64}$$

From the aggregate price index follows that the average price in period t is:

$$\begin{aligned} P_t^{m,c} &= \left[\int_0^1 (P_{it}^{m,c})^{1-\eta_t^{m,c}}\,di\right]^{\frac{1}{1-\eta_t^{m,c}}} \\ &= \left[\left(\int_0^{\xi^m}\left(P_{t-1}^{m,c}\pi_{t-1}^{m,c}\right)^{1-\eta_t^{m,c}} + \int_{\xi^m}^1\left(P_{new,t}^{m,c}\right)^{1-\eta_t^{m,c}}\right)di\right]^{\frac{1}{1-\eta_t^{m,c}}} \\ &= \left[\xi^m\left(P_{t-1}^{m,c}\pi_{t-1}^{m,c}\right)^{1-\eta_t^{m,c}} + (1-\xi^m)\left(P_{new,t}^{m,c}\right)^{1-\eta_t^{m,c}}\right]^{\frac{1}{1-\eta_t^{m,c}}}. \end{aligned} \tag{65}$$

Dividing:

$$1 = \left[\xi^m\left(\frac{\pi_{t-1}^{m,c}}{\pi_t^{m,c}}\right)^{1-\eta_t^{m,c}} + (1-\xi^m)\left(\frac{P_{new,t}^{m,c}}{P_t^{m,c}}\right)^{1-\eta_t^{m,c}}\right]^{\frac{1}{1-\eta_t^{m,c}}}. \tag{66}$$

Linearize equation (66). Start by taking both sides to the power of $1-\eta_t^{m,c}$:

$$D = \left[\xi^m \left(\frac{\pi_{t-1}^{m,c}}{\pi_t^{m,c}} \right)^{1-\eta_t^{m,c}} + (1-\xi^m)(\widetilde{p}_t^{m,c})^{1-\eta_t^{m,c}} \right] = 1. \tag{67}$$

Differentiating this and evaluating in steady state:

$$\frac{\partial D}{\partial \widetilde{p}_t^{m,c}} = (1-\eta^{m,c})(1-\xi^m). \tag{68}$$

$$\frac{\partial D}{\partial \pi_{t-1}^{m,c}} = (1-\eta^{m,c})\xi^m \frac{1}{\overline{\pi}}. \tag{69}$$

$$\frac{\partial D}{\partial \pi_t^{m,c}} = -(1-\eta^{m,c})\xi^m \frac{1}{\overline{\pi}}. \tag{70}$$

Differentiating with respect to $\eta_t^{m,c}$ we require the derivative with respect to a power. Note that

$$f(\eta_t^{m,c}) = x^{1-\eta_t^{m,c}} = e^{\ln\left(x^{1-\eta_t^{m,c}}\right)} = e^{(1-\eta_t^{m,c})\ln x}.$$

Then, differentiating this expression with respect to $\eta_t^{m,c}$:

$$\begin{aligned} \frac{\partial f}{\partial \eta_t^{m,c}} &= e^{(1-\eta_t^{m,c})(\ln x)}(-\ln x) \\ &= -(\ln x)\, x^{1-\eta_t^{m,c}}. \end{aligned}$$

It is important to note, that x is defined as; $x = \tilde{p}^{m,c}$ and $x = \overline{\pi}/\overline{\pi}$ in equation (67). In each case, $x = 1$ in steady state. Note also that $\ln(1) = 0$, so $\widehat{\eta}_t^{m,c}$ does not appear in the linear expansion of (67).

Collecting terms and rearranging:

$$\begin{aligned} 0 &= dD = (1-\eta^{m,c})(1-\xi^m)\widetilde{p}^{m,c}\frac{d\widetilde{p}_t^{m,c}}{\widetilde{p}^{m,c}} \\ &\quad + (1-\eta^{m,c})\xi^m \frac{1}{\overline{\pi}}\overline{\pi}\frac{d\pi_{t-1}^{m,c}}{\overline{\pi}} - (1-\eta^{m,c})\xi^m \frac{1}{\overline{\pi}}\overline{\pi}\frac{d\pi_t^{m,c}}{\overline{\pi}}. \end{aligned}$$

or,

$$(1-\xi^m)\widehat{\widehat{p}}_t^{m,c}+\xi^m\left(\widehat{\pi}_{t-1}^{m,c}-\widehat{\pi}_t^{m,c}\right)=0.$$

Solving for $\widehat{\widehat{p}}_t$:

$$\widehat{\widehat{p}}_t=\frac{\xi^m}{(1-\xi^m)}\left(\widehat{\pi}_t^{m,c}-\widehat{\pi}_{t-1}^{m,c}\right)=0. \tag{71}$$

Inserting equation (71) in equation (64):

$$\begin{aligned}&\frac{\xi^m}{(1-\xi^m)}\left(\widehat{\pi}_t^{m,c}-\widehat{\pi}_{t-1}^{m,c}\right)=-\beta\xi^m\widehat{\pi}_t^{m,c}\\&+(1-\beta\xi^m)\sum_{j=1}^{\infty}(\beta\xi^m)^j\,\widehat{\pi}_{t+j}^{m,c}+(1-\beta\xi^m)\sum_{j=0}^{\infty}(\beta\xi^m)^j\left(\widehat{mc}_{t+j}^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_{t+j}^{m,c}\right).\end{aligned}$$

Simplifying and rearranging:

$$\begin{aligned}&[1+(1-\xi^m)\beta]\,\widehat{\pi}_t^{m,c}-\widehat{\pi}_{t-1}^{m,c}\\=\;&\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\sum_{j=1}^{\infty}(\beta\xi^m)^j\,\widehat{\pi}_{t+j}^{m,c}\\&+\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\sum_{j=0}^{\infty}(\beta\xi^m)^j\left(\widehat{mc}_{t+j}^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_{t+j}^{m,c}\right).\end{aligned} \tag{72}$$

Lead this expression forward by one period and multiply by $\beta\xi^m$:

$$\begin{aligned}&\beta\xi^m\,[1+(1-\xi^m)\beta]\,\widehat{\pi}_{t+1}^{m,c}-\beta\xi^m\widehat{\pi}_t^{m,c}\\=\;&\frac{\beta\xi^m(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\sum_{j=1}^{\infty}(\beta\xi^m)^j\,\widehat{\pi}_{t+j+1}^{m,c}\\&+\frac{\beta\xi^m(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\sum_{j=0}^{\infty}(\beta\xi^m)^j\left(\widehat{mc}_{t+j+1}^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_{t+j+1}^{m,c}\right).\end{aligned}$$

$$\begin{aligned}&\beta\xi^m\,[1+(1-\xi^m)\beta]\,\widehat{\pi}_{t+1}^{m,c}-\beta\xi^m\widehat{\pi}_t^{m,c}\\=\;&\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\sum_{j=2}^{\infty}(\beta\xi^m)^j\,\widehat{\pi}_{t+j}^{m,c}\\&+\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\sum_{j=1}^{\infty}(\beta\xi^m)^j\left(\widehat{mc}_{t+j}^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_{t+j}^{m,c}\right).\end{aligned} \tag{73}$$

Subtract equation (73) from equation (72), to obtain:

$$\begin{aligned} & \left[1+(1-\xi^m)\beta+\beta\xi^m\right]\widehat{\pi}_t^{m,c}-\beta\xi^m\left[1+(1-\xi^m)\beta\right]\widehat{\pi}_{t+1}^{m,c}-\widehat{\pi}_{t-1}^{m,c} \\ = \quad & \frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\left(\beta\xi^m\right)\widehat{\pi}_{t+1}^{m,c} \\ & +\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\left(\beta\xi^m\right)^0\left(\widehat{mc}_t^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_t^{m,c}\right). \end{aligned} \tag{74}$$

Combining terms:

$$\begin{aligned} & \left[1+(1-\xi^m)\beta+\beta\xi^m\right]\widehat{\pi}_t^{m,c}-\widehat{\pi}_{t-1}^{m,c} \\ = \quad & \beta\xi^m\left[1+(1-\xi^m)\beta\right]\widehat{\pi}_{t+1}^{m,c}+(1-\xi^m)(1-\beta\xi^m)\beta\widehat{\pi}_{t+1}^{m,c} \\ & +\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\left(\widehat{mc}_t^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_t^{m,c}\right). \end{aligned} \tag{75}$$

Note that we can simplify using the following;

$$\begin{aligned} & \beta\xi^m+\beta\xi^m(1-\xi^m)\beta+(1-\xi^m)(1-\beta\xi^m)\beta \\ = \quad & \left[\frac{\beta\xi^m}{\beta}+\beta\xi^m(1-\xi^m)\right]\beta+\left[(1-\xi^m)-\beta\xi^m(1-\xi^m)\right]\beta \\ = \quad & \left[\xi^m+(1-\xi^m)\right]\beta \\ = \quad & \beta. \end{aligned}$$

Thus,

$$[1+\beta]\,\widehat{\pi}_t^{m,c}-\beta\widehat{\pi}_{t+1}^{m,c}-\widehat{\pi}_{t-1}^{m,c}=\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m}\left(\widehat{mc}_t^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_t^{m,c}\right). \tag{76}$$

or,

$$\widehat{\pi}_t^{m,c}=\frac{\beta}{1+\beta}\mathrm{E}_t\widehat{\pi}_{t+1}^{m,c}+\frac{1}{1+\beta}\widehat{\pi}_{t-1}^{m,c}+\frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m(1+\beta)}\left(\widehat{mc}_t^{m,c}-\frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_t^{m,c}\right). \tag{77}$$

2.1.2 Log-linearizing the imported pricing equation (investment good)

Equivalently, the log-linearized pricing equation for the imported investment goods follows:

$$\widehat{\pi}_t^{m,i} = \frac{\beta}{1+\beta}\mathrm{E}_t\widehat{\pi}_{t+1}^{m,i} + \frac{1}{1+\beta}\widehat{\pi}_{t-1}^{m,i} + \frac{(1-\xi^m)(1-\beta\xi^m)}{\xi^m(1+\beta)}\left(\widehat{mc}_t^{m,i} - \frac{1}{(\eta^{m,c}-1)^2}\widehat{\eta}_t^{m,i}\right). \tag{78}$$

2.2 Domestic price determination using Calvo price setting

As above, each domestic firm faces a random probability $(1-\xi^d)$ that he/she can reoptimize her price in any period. Since all firms are equal (facing the same Calvo probability of a price change) the subscript i can be dropped. The reoptimized price is denoted $P_{new,t}$. With probability ξ^d the firm does not reoptimize, and its price in period $t+j$ is $(\pi_t\pi_{t+1}...\pi_{t+j-1})\,P_{new,t}$. The firm faces the following optimization problem:

$$\max_{P_{new,t}} \ \mathrm{E}_t\sum_{j=0}^{\infty}\left(\beta\xi^d\right)^j \upsilon_{t+j}[((\pi_t\pi_{t+1}...\pi_{t+j-1})\,P_{new,t})\left(\frac{(\pi_t\pi_{t+1}...\pi_{t+j-1}^j)P_{new,t}}{P_{t+j}}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j}$$
$$-MC_{t+j}\left(\frac{(\pi_t\pi_{t+1}...\pi_{t+j-1}^j)P_{new,t}}{P_{t+j}}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j} - MC_{t+j}z_{t+j}\phi]. \tag{79}$$

Note that the fixed cost $z_t\phi$ (denoted in real terms) enters the production function in order to make profits zero in steady state. However, we do not want this to have an affect on the pricing decision why it is translated into nominal terms in the profit maximization by MC_t (rather than the price, $P_{new,t}$).

The first order condition yields:

$$\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta\xi^d\right)^j \upsilon_{t+j}[\tfrac{-1}{\lambda_{f,t}-1} \left(P_{new,t}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \left(\pi_t\pi_{t+1}...\pi_{t+j-1}^j\right)^{1-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \left(P_{t+j}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j}$$
$$+\tfrac{\lambda_{f,t}}{\lambda_{f,t}-1} \left(P_{new,t}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}-1} \left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \left(P_{t+j}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j} MC_{t+j}] = 0. \tag{80}$$

$$\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta\xi^d\right)^j \upsilon_{t+j} \left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \left(P_{t+j}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j} \times$$
$$[\tfrac{-1}{\lambda_{f,t}-1} \left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right) \left(P_{new,t}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} + \tfrac{\lambda_{f,t}}{\lambda_{f,t}-1} \left(P_{new,t}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}-1} MC_{t+j}] = 0. \tag{81}$$

Multiply and divide by P_{t+j}:

$$\begin{gathered}\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta\xi^d\right)^j \upsilon_{t+j} \left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \left(P_{t+j}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j} P_{t+j} \times \\ \left[\tfrac{\left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)P_{new,t}}{P_{t+j}} - \tfrac{\lambda_{f,t} MC_{t+j}}{P_{t+j}}\right] = 0.\end{gathered} \tag{82}$$

Use the fact that $P_{t+j} \equiv P_t\pi_{t+1}\pi_{t+2}...\pi_{t+j}$

$$\begin{gathered}\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta\xi^d\right)^j \upsilon_{t+j} \left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \left(P_t\pi_{t+1}\pi_{t+2}...\pi_{t+j}\right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j} P_{t+j} \times \\ \left[\tfrac{\left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)P_{new,t}}{P_t\pi_{t+1}\pi_{t+2}...\pi_{t+j}} - \tfrac{\lambda_{f,t} MC_{t+j}}{P_{t+j}}\right] = 0.\end{gathered} \tag{83}$$

Multiply through with $\left(P_t\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}}$. This implies:

$$\begin{gathered}E_t \sum_{j=0}^{\infty} \left(\beta\xi^d\right)^j \upsilon_{t+j} \left(\tfrac{\left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)}{\left(\pi_{t+1}\pi_{t+2}...\pi_{t+j}\right)}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} Y_{t+j} P_{t+j} \times \\ \left[\tfrac{\left(\pi_t\pi_{t+1}...\pi_{t+j-1}\right)}{\left(\pi_{t+1}\pi_{t+2}...\pi_{t+j}\right)} \tfrac{P_{new,t}}{P_t} - \tfrac{\lambda_{f,t} MC_{t+j}}{P_{t+j}}\right] = 0\end{gathered} \tag{84}$$

Rearrange and stationarize with the technology shock z_t:

$$E_t \sum_{j=0}^{\infty} \left(\beta\xi^d\right)^j \psi_{t+j} \left(X_{t,j}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} z_{t+j} Y_{t+j}^{\#} \left[X_{t,j} \frac{P_{new,t}}{P_t} - \frac{\lambda_{f,t} MC_{t+j}}{P_{t+j}}\right] = 0, \tag{85}$$

$$F = \mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta \xi^d\right)^j \psi_{z,t+j} \left(X_{t,j}\right)^{-\frac{\lambda_{f,t}}{\lambda_{f,t-1}}} Y^{\#}_{t+j} \left[X_{t,j} \widetilde{p}_t - \lambda_{f,t} mc_{t+j}\right] = 0, \quad (86)$$

where $\psi_{t+j} \equiv \upsilon_{t+j} P_{t+j}$, $X_{t,j} \equiv \frac{(\pi_t \pi_{t+1} \ldots \pi_{t+j-1})}{(\pi_{t+1} \pi_{t+2} \ldots \pi_{t+j})} = \frac{\pi_t}{\pi_{t+j}}$, and$Y^{\#}_{t+j} = \frac{Y_{t+j}}{z_{t+j}}.\psi_{z,t} \equiv \upsilon_t P_t z_t$, $\widetilde{p}_t = \frac{P_{new,t}}{P_t}$, and $mc_t = \frac{MC_t}{P_t}$.

2.2.1 Log-linearizing the domestic price equation

We will use a first order Taylor approximation around the steady state to log-linearize the price equation. This is obtained by totally differentiating equation (86) and evaluating it in steady-state:[2]

$$dF = F_{\widetilde{p}} d\widetilde{p}_t + F_{mc} dmc_{t+j} + F_{\pi} d\pi_{t+j} + F_{\pi} d\pi_t + F_{\lambda_f} d\lambda_{f,t+j}. \quad (87)$$

$$\frac{dF}{F} = F_{\widetilde{p}} d\widetilde{p}_t \frac{\widetilde{p}}{\widetilde{p}F} + F_{mc} dmc_{t+j} \frac{mc}{mcF} + F_{\pi} d\pi_{t+j} \frac{\pi}{\pi F} + F_{\pi} d\pi_t \frac{\pi}{\pi F} + F_{\lambda_f} d\lambda_{f,t+j} \frac{\lambda_f}{\lambda_f F}. \quad (88)$$

$$\widetilde{p} F_{\widetilde{p}} \frac{d\widetilde{p}_t}{\widetilde{p}} + mc F_{mc} \frac{dmc_{t+j}}{mc} + \pi F_{\pi} \frac{d\pi_{t+j}}{\pi} + \pi F_{\pi} \frac{d\pi_t}{\pi} + \lambda_f F_{\lambda_f} \frac{d\lambda_{f,t+j}}{\lambda_f} = 0. \quad (89)$$

$$\widetilde{p} F_{\widetilde{p}} \widehat{\widetilde{p}}_t + mc F_{mc} \widehat{mc}_{t+j} + \pi F_{\pi} \widehat{\pi}_{t+j} + \pi F_{\pi} \widehat{\pi}_t + \lambda_f F_{\lambda_f} \widehat{\lambda}_{f,t+j} = 0, \quad (90)$$

where a hat denotes the log deviation from steady state (i.e., $x_t = \frac{dx_t}{x} = \ln X_t - \ln X$). Note that in steady-state; $Y^{\#}_{t+j}$ and $\psi_{z,t+j}$ are constants, $Y^{\#}$ and ψ_z,respectively, $\widetilde{p} = \lambda_f mc = 1, \pi_t = \overline{\pi}$, and $X_{t,j} = \frac{\overline{\pi}}{\overline{\pi}} = 1$.

[2] To see this, note the following. A first order Taylor approximation of $G(X_t, Y_t) = \ln F(X_t, Y_t) = \ln F(e^{\ln X_t}, e^{\ln Y_t})$ around the steady state implies:

$G \simeq G(e^{\ln X}, e^{\ln Y}) + G_X(\ln X_t - \ln X) + G_Y(\ln Y_t - \ln Y)$,

$\ln\left(\frac{F(X_t,Y_t)}{F(X,Y)}\right) \simeq \frac{1}{F(X,Y)} F_x X(\ln X_t - \ln X) + \frac{1}{F(X,Y)} F_Y Y(\ln Y_t - \ln Y)$.

This is equivalent to: $\frac{dF(X_t,Y_t)}{F(X,Y)} \simeq \frac{1}{F(X,Y)} F_x X \frac{dX_t}{X} + \frac{1}{F(X,Y)} F_Y Y \frac{dY_t}{Y}$.

Start by differentiating equation (86) with respect to the variables $\widetilde{p}_t$, s_{t+j}, π_{t+j}, π_t and $\lambda_{f,t+j}$, evaluating all the first order conditions in steady-state:

$$\begin{aligned} \frac{\partial F}{\partial \widetilde{p}_t} &= \sum_{j=0}^{\infty} \left(\beta \xi^d\right)^j \psi_z Y^{\#} \\ &= \frac{1}{(1-\beta\xi^d)} \psi_z Y^{\#}. \end{aligned} \tag{91}$$

$$\frac{\partial F}{\partial mc_{t+j}} = -\left(\beta \xi^d\right)^j \psi_z Y^{\#} \lambda_f, \qquad \forall \quad j \geq 0,$$

i.e.,

$$\frac{\partial F}{\partial mc_{t+j}} = -\sum_{j=0}^{\infty} \left(\beta \xi^d\right)^j \psi_z Y^{\#} \lambda_f. \tag{92}$$

Regarding the derivatives with respect to π_t and π_{t+j}, note that $\widetilde{p} = \lambda_f mc$ in steady state why we only have to differentiate with respect to the $\pi-$terms within brackets.

$$\begin{aligned} \frac{\partial F}{\partial \pi_t} &= \sum_{j=1}^{\infty} \left(\beta \xi^d\right)^j \psi_z Y^{\#} \left(\frac{\widetilde{p}}{\overline{\pi}}\right) \\ &= \frac{\beta\xi^d}{(1-\beta\xi^d)} \psi_z Y^{\#} \left(\frac{1}{\overline{\pi}}\right). \end{aligned} \tag{93}$$

$$\frac{\partial F}{\partial \pi_{t+j}} = \left(\beta \xi^d\right)^j \psi_z Y^{\#} \widetilde{p} \left(\frac{-1}{\overline{\pi}}\right), \qquad \forall \quad j > 0,$$

i.e.,

$$\frac{\partial F}{\partial \pi_{t+j}} = \sum_{j=1}^{\infty} \left(\beta \xi^d\right)^j \psi_z Y^{\#} \left(\frac{-1}{\overline{\pi}}\right). \tag{94}$$

Note again that $\widetilde{p} - \lambda_f mc$ is zero in steady state why we only have to differentiate with respect to the $\lambda_{f,t+j}$-terms within brackets.

$$\frac{\partial F}{\partial \lambda_{f,t+j}} = -\left(\beta \xi^d\right)^j \psi_z Y^{\#} mc, \qquad \forall \quad j \geq 0,$$

i.e.,

$$\frac{\partial F}{\partial \lambda_{f,t+j}} = -\sum_{j=0}^{\infty} \left(\beta \xi^d\right)^j \psi_z Y^{\#} mc. \tag{95}$$

Collecting terms and inserting in equation (89):

$$\begin{aligned}
0 &= \frac{1}{(1-\beta\xi^{d})}\psi_z Y^{\#}\widetilde{p}\frac{d\widetilde{p}_t}{\widetilde{p}} + \frac{\beta\xi^{d}}{(1-\beta\xi^{d})}\psi_z Y^{\#}\left(\frac{1}{\overline{\pi}}\right)\overline{\pi}\frac{d\pi_t}{\overline{\pi}} \\
&\quad -\sum_{j=1}^{\infty}\left(\beta\xi^{d}\right)^{j}\psi_z Y^{\#}\left(\frac{1}{\overline{\pi}}\right)\overline{\pi}\frac{d\pi_{t+j}}{\overline{\pi}} - \sum_{j=0}^{\infty}\left(\beta\xi^{d}\right)^{j}\psi_z Y^{\#}\lambda_f mc\frac{dmc_{t+j}}{mc} \\
&\quad -\sum_{j=0}^{\infty}\left(\beta\xi^{d}\right)^{j}\psi_z Y^{\#}\lambda_f mc\frac{d\lambda_{f,t+j}}{\lambda_f}. \qquad (96)
\end{aligned}$$

Taking into account that $\lambda_f mc = \widetilde{p} = 1$ and rearranging, the Taylor expansion is, consequently;

$$\begin{aligned}
0 &= \frac{\psi_z Y^{\#}}{(1-\beta\xi^{d})}\widehat{\widetilde{p}}_t + \frac{\beta\xi^{d}\psi_z Y^{\#}}{(1-\beta\xi^{d})}\widehat{\pi}_t - \sum_{j=1}^{\infty}\left(\beta\xi^{d}\right)^{j}\psi_z Y^{\#}\widehat{\pi}_{t+j} \\
&\quad -\sum_{j=0}^{\infty}\left(\beta\xi^{d}\right)^{j}\psi_z Y^{\#}\left(\widehat{mc}_{t+j}+\widehat{\lambda}_{f,t+j}\right). \qquad (97)
\end{aligned}$$

Solving for $\widehat{\widetilde{p}}_t$

$$\begin{aligned}
\widehat{\widetilde{p}}_t &= \frac{(1-\beta\xi^{d})}{\psi_z Y^{\#}}[\frac{(-\beta\xi^{d})\psi_z Y^{\#}}{(1-\beta\xi^{d})}\widehat{\pi}_t + \sum_{j=1}^{\infty}\left(\beta\xi^{d}\right)^{j}\psi_z Y^{\#}\widehat{\pi}_{t+j} \\
&\quad +\sum_{j=0}^{\infty}\left(\beta\xi^{d}\right)^{j}\psi_z Y^{\#}\left(\widehat{mc}_{t+j}+\widehat{\lambda}_{f,t+j}\right)]. \qquad (98)
\end{aligned}$$

$$\widehat{\widetilde{p}}_t = -\beta\xi^{d}\widehat{\pi}_t + (1-\beta\xi^{d})\sum_{j=1}^{\infty}\left(\beta\xi^{d}\right)^{j}\widehat{\pi}_{t+j} + (1-\beta\xi^{d})\sum_{j=0}^{\infty}\left(\beta\xi^{d}\right)^{j}\left(\widehat{mc}_{t+j}+\widehat{\lambda}_{f,t+j}\right). \qquad (99)$$

From the aggregate price index follows that the average price in period t is:

$$\begin{aligned}
P_t &= \left[\left(\int_0^{\xi^{d}}\left(P_{t-1}\pi_{t-1}\right)^{\frac{1}{1-\lambda_{f,t}}} + \int_{\xi^{d}}^{1}\left(P_{new,t}\right)^{\frac{1}{1-\lambda_{f,t}}}\right)di\right]^{1-\lambda_{f,t}} \\
&= \left[\xi^{d}\left(P_{t-1}\pi_{t-1}\right)^{\frac{1}{1-\lambda_{f,t}}} + (1-\xi^{d})\left(P_{new,t}\right)^{\frac{1}{1-\lambda_{f,t}}}\right]^{1-\lambda_{f,t}}. \qquad (100)
\end{aligned}$$

Dividing:

$$1 = \left[\xi^d \left(\frac{\pi_{t-1}}{\pi_t}\right)^{\frac{1}{1-\lambda_{f,t}}} + (1-\xi^d)\left(\frac{P_{new,t}}{P_t}\right)^{\frac{1}{1-\lambda_{f,t}}}\right]^{1-\lambda_{f,t}}. \tag{101}$$

Linearize equation (101). Start by taking both sides to the power of $\frac{1}{1-\lambda_{f,t}}$:

$$D = \left[\xi^d \left(\frac{\pi_{t-1}}{\pi_t}\right)^{\frac{1}{1-\lambda_{f,t}}} + (1-\xi^d)\left(\frac{P_{new,t}}{P_t}\right)^{\frac{1}{1-\lambda_{f,t}}}\right] = 1. \tag{102}$$

Differentiating this and evaluating in steady state:

$$\frac{\partial D}{\partial \widetilde{p}_t} = \frac{(1-\xi^d)}{(1-\lambda_f)}. \tag{103}$$

$$\frac{\partial D}{\partial \pi_{t-1}} = \frac{\xi^d}{(1-\lambda_f)}\frac{1}{\overline{\pi}}. \tag{104}$$

$$\frac{\partial D}{\partial \pi_t} = \frac{-\xi^d}{(1-\lambda_f)}\frac{1}{\overline{\pi}}. \tag{105}$$

Differentiating with respect to $\lambda_{f,t}$ we require the derivative with respect to a power. Note that

$$f(\lambda_{f,t}) = x^{\frac{1}{1-\lambda_{f,t}}} = e^{\ln\left(x^{\frac{1}{1-\lambda_{f,t}}}\right)} = e^{\frac{1}{1-\lambda_{f,t}}\ln x}.$$

Then, differentiating this expression with respect to $\lambda_{f,t}$:

$$\begin{aligned}\frac{\partial f}{\partial \lambda_{f,t}} &= e^{\frac{1}{1-\lambda_{f,t}}\ln x}(\ln x)\left(\frac{-1}{(1-\lambda_{f,t})^2}(-1)\right)\\ &= (\ln x)\, x^{\frac{1}{1-\lambda_{f,t}}}\left(\frac{1}{(1-\lambda_{f,t})^2}\right).\end{aligned}$$

It is important to note, that x is defined as; $x = \tilde{p}$ and $x = \overline{\pi}/\overline{\pi}$ in equation (102). In each case, $x = 1$ in steady state. Note also that $\ln(1) = 0$, so $\hat{\lambda}_{f,t}$ does not appear in the linear expansion of (102). Consequently, collecting terms and rearranging:

$$dD = \frac{(1-\xi^d)}{(1-\lambda_f)}\widetilde{p}\frac{d\widetilde{p}_t}{\widetilde{p}} + \frac{\xi^d}{(1-\lambda_f)}\frac{1}{\pi}\pi\frac{d\pi_{t-1}}{\pi} - \frac{\xi^d}{(1-\lambda_f)}\frac{1}{\pi}\pi\frac{d\pi_t}{\pi} = 0.$$

or,

$$(1-\xi^d)\widehat{\widetilde{p}}_t + \xi^d\left(\widehat{\pi}_{t-1} - \widehat{\pi}_t\right) = 0.$$

Solving for $\widehat{\widetilde{p}}_t$:

$$\widehat{\widetilde{p}}_t = \frac{\xi^d}{(1-\xi^d)}\left(\widehat{\pi}_t - \widehat{\pi}_{t-1}\right) = 0. \tag{106}$$

Inserting equation (106) in equation (99):

$$\frac{\xi^d}{(1-\xi^d)}\left(\widehat{\pi}_t - \widehat{\pi}_{t-1}\right) = -\beta\xi^d\widehat{\pi}_t + (1-\beta\xi^d)\sum_{j=1}^{\infty}\left(\beta\xi^d\right)^j\widehat{\pi}_{t+j}$$
$$+(1-\beta\xi^d)\sum_{j=0}^{\infty}\left(\beta\xi^d\right)^j\left(\widehat{mc}_{t+j} + \widehat{\lambda}_{f,t+j}\right).$$

Simplifying:

$$\left[1+(1-\xi^d)\beta\right]\widehat{\pi}_t - \widehat{\pi}_{t-1} = \frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\sum_{j=1}^{\infty}\left(\beta\xi^d\right)^j\widehat{\pi}_{t+j} \tag{107}$$
$$+\frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\sum_{j=0}^{\infty}\left(\beta\xi^d\right)^j\left(\widehat{mc}_{t+j} + \widehat{\lambda}_{f,t+j}\right).$$

Lead this expression forward by one period and multiply by $\beta\xi^d$:

$$\beta\xi^d\left[1+(1-\xi^d)\beta\right]\widehat{\pi}_{t+1} - \beta\xi^d\widehat{\pi}_t = \frac{\beta\xi^d(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\sum_{j=1}^{\infty}\left(\beta\xi^d\right)^j\widehat{\pi}_{t+j+1}$$
$$+\frac{\beta\xi^d(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\sum_{j=0}^{\infty}\left(\beta\xi^d\right)^j\left(\widehat{mc}_{t+j+1} + \widehat{\lambda}_{f,t+j+1}\right),$$

$$\beta\xi^d\left[1+(1-\xi^d)\beta\right]\widehat{\pi}_{t+1} - \beta\xi^d\widehat{\pi}_t = \frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\sum_{j=2}^{\infty}\left(\beta\xi^d\right)^j\widehat{\pi}_{t+j} \tag{108}$$
$$+\frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\sum_{j=1}^{\infty}\left(\beta\xi^d\right)^j\left(\widehat{mc}_{t+j} + \widehat{\lambda}_{f,t+j}\right).$$

Subtract equation (108) from equation (107), to obtain:

$$
\begin{aligned}
&\left[1+(1-\xi^d)\beta+\beta\xi^d\right]\widehat{\pi}_t-\beta\xi^d\left[1+(1-\xi^d)\beta\right]\widehat{\pi}_{t+1}-\widehat{\pi}_{t-1} \\
= \;& \frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\left(\beta\xi^d\right)\widehat{\pi}_{t+1}+\frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}(\beta\xi)^0\left(\widehat{mc}_t+\widehat{\lambda}_{f,t}\right)
\end{aligned} \tag{109}
$$

Combining terms:

$$
\begin{aligned}
&\left[1+(1-\xi^d)\beta+\beta\xi^d\right]\widehat{\pi}_t-\widehat{\pi}_{t-1} \\
= \;& \beta\xi^d\left[1+(1-\xi^d)\beta\right]\widehat{\pi}_{t+1}+(1-\xi^d)(1-\beta\xi^d)\beta\widehat{\pi}_{t+1}+\frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\left(\widehat{mc}_t+\widehat{\lambda}_{f,t}\right),
\end{aligned} \tag{110}
$$

$$
[1+\beta]\,\widehat{\pi}_t-\beta\widehat{\pi}_{t+1}-\widehat{\pi}_{t-1}=\frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d}\left(\widehat{mc}_t+\widehat{\lambda}_{f,t}\right), \tag{111}
$$

or,

$$
\widehat{\pi}_t=\frac{\beta}{1+\beta}\mathrm{E}_t\widehat{\pi}_{t+1}+\frac{1}{1+\beta}\widehat{\pi}_{t-1}+\frac{(1-\xi^d)(1-\beta\xi^d)}{\xi^d(1+\beta)}\left(\widehat{mc}_t+\widehat{\lambda}_{f,t}\right). \tag{112}
$$

2.3 Export price determination using Calvo price setting

The exporting firms buy the final domestic good and differentiates it by brand naming. Subsequently they sell the (continuum of) differentiated goods to the households in the foreign market. The marginal cost is the input price $MC_t^{df} = P_t^d$. The exporting firms face the following demand for each product i:

$$
X_{i,t}^{df}=\left(\frac{P_{i,t}^{df}}{P_t^{df}}\right)^{-\frac{\lambda_{x,t}}{\lambda_{xt}-1}}X_t^{df}, \tag{113}
$$

where it is assumed that the export prices are invoiced in the local currency of the export market. Note that we allow for different elasticities between the differentiated goods abroad and at home. The steady state markup thus differs between the domestic and the export market (λ_f and λ_x, respectively). Note also that the steady state markup is one in the export market;

$$P^{df} = \frac{P}{S}. \tag{114}$$

Furthermore we assume that prices are sticky in the foreign currency, following the Calvo model, so that there will be incomplete exchange rate pass-through in the export market. Each export firm faces a random probability $(1 - \xi^{df})$ that he/she can reoptimize her local currency price in any period. Since all firms are equal (facing the same Calvo probability of a price change) the subscript i can be dropped. The reoptimized price (in foreign currency) is denoted $P^{df}_{new,t}$. With probability ξ^{df} the export firm does not reoptimize, and its price in period $t + j$ is $\left(\pi^{df}_t \pi^{df}_{t+1} ... \pi^{df}_{t+j-1}\right) P^{df}_{new,t}$. The export firms maximize profits (denoted in the local currency) taking into account that there might not be a chance to optimally change the price:

$$\max_{P^{df}_{new,t}} \ \mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta\xi^{df}\right)^j \upsilon_{t+j} [\left(\left(\pi^{df}_t \pi^{df}_{t+1} ... \pi^{df}_{t+j-1}\right) P^{df}_{new,t}\right) \left(\frac{\left(\pi^{df}_t \pi^{df}_{t+1} ... \pi^{df}_{t+j-1}\right) P^{df}_{new,t}}{P^{df}_{t+j}}\right)^{-\frac{\lambda_{x,t}}{\lambda_{xt}-1}} X^{df}_{t+j} - \frac{MC^{df}_{t+j}}{S_{t+j}} \left(\frac{\left(\pi^{df}_t \pi^{df}_{t+1} ... \pi^{df}_{t+j-1}\right) P^{df}_{new,t}}{P^{df}_{t+j}}\right)^{-\frac{\lambda_{x,t}}{\lambda_{xt}-1}} X^{df}_{t+j} - \frac{MC^{df}_{t+j}}{S_{t+j}} z_{t+j} \phi^{df}]. \tag{115}$$

Note that we assume that the stochastic discount factor $\beta\upsilon_{t+j}$ is the same for the exporting firms as the domestic firms. The first order condition yields:

$$\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta\xi^{df}\right)^j \upsilon_{t+j} [\frac{-1}{\lambda_{x,t}-1} \left(P^{df}_{new,t}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t}-1}} \left(\pi^{df}_t \pi^{df}_{t+1} ... \pi^{df}_{t+j-1}\right)^{1-\frac{\lambda_{x,t}}{\lambda_{xt}-1}} \left(P^{df}_{t+j}\right)^{\frac{\lambda_{x,t}}{\lambda_{xt}-1}} X^{df}_{t+j} + \frac{\lambda_{x,t}}{\lambda_{xt}-1} \left(P^{df}_{new,t}\right)^{-\frac{\lambda_{x,t}}{\lambda_{xt}-1}-1} \left(\pi^{df}_t \pi^{df}_{t+1} ... \pi^{df}_{t+j-1}\right)^{-\frac{\lambda_{x,t}}{\lambda_{xt}-1}} \left(P^{df}_{t+j}\right)^{\frac{\lambda_{x,t}}{\lambda_{xt}-1}} X^{df}_{t+j} \frac{MC^{df}_{t+j}}{S_{t+j}}] = 0. \tag{116}$$

$$\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta \xi^{df}\right)^j \upsilon_{t+j} \left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} \left(P_{t+j}^{df}\right)^{\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} X_{t+j}^{df} \times$$
$$[\tfrac{-1}{\lambda_{x,t}-1} \left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right) \left(P_{new,t}^{df}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} + \tfrac{\lambda_{x,t}}{\lambda_{x,t}-1} \left(P_{new,t}^{df}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}-1} \tfrac{MC_{t+j}^{df}}{S_{t+j}}] = 0. \tag{117}$$

Multiply and divide by P_{t+j}^{df}:

$$\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta \xi^{df}\right)^j \upsilon_{t+j} \left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} \left(P_{t+j}^{df}\right)^{\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} X_{t+j}^{df} P_{t+j}^{df} \times$$
$$\left[\tfrac{\left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right) P_{new,t}^{df}}{P_{t+j}^{df}} - \tfrac{\lambda_{x,t} MC_{t+j}}{S_{t+j} P_{t+j}^{df}}\right] = 0. \tag{118}$$

Use the fact that $P_{t+j}^{df} \equiv P_t^{df} \pi_{t+1}^{df} \pi_{t+2}^{df} ... \pi_{t+j}^{df}$

$$\mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta \xi^{df}\right)^j \upsilon_{t+j} \left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} \left(P_t^{df} \pi_{t+1}^{df} \pi_{t+2}^{df} ... \pi_{t+j}^{df}\right)^{\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} X_{t+j}^{df} P_{t+j}^{df} \times$$
$$\left[\tfrac{\left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right) P_{new,t}^{df}}{P_t^{df} \pi_{t+1}^{df} \pi_{t+2}^{df} ... \pi_{t+j}^{df}} - \tfrac{\lambda_{x,t} MC_{t+j}^{df}}{S_{t+j} P_{t+j}^{df}}\right] = 0. \tag{119}$$

Multiply through with $\left(P_t^{df}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}}$. This implies:

$$E_t \sum_{j=0}^{\infty} \left(\beta \xi^{df}\right)^j \upsilon_{t+j} \left(\tfrac{\left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right)}{\left(\pi_{t+1}^{df} \pi_{t+2}^{df} ... \pi_{t+j}^{df}\right)}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} X_{t+j}^{df} P_{t+j}^{df} \times \left[\tfrac{\left(\pi_t^{df} \pi_{t+1}^{df} ... \pi_{t+j-1}^{df}\right)}{\left(\pi_{t+1}^{df} \pi_{t+2}^{df} ... \pi_{t+j}^{df}\right)} \tfrac{P_{new,t}^{df}}{P_t^{df}} - \tfrac{\lambda_{x,t} MC_{t+j}^{df}}{S_{t+j} P_{t+j}^{df}}\right] = 0 \tag{120}$$

Rearrange and stationarize with the technology shock z_{t+j}^{*}:

$$E_t \sum_{j=0}^{\infty} \left(\beta \xi^{df}\right)^j \psi_{t+j}^{df} \left(\frac{\pi_t^{df}}{\pi_{t+j}^{df}}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} z_{t+j}^{*} X_{t+j}^{\#df} \left[\frac{\pi_t^{df}}{\pi_{t+j}^{df}} \frac{P_{new,t}^{df}}{P_t^{df}} - \frac{\lambda_{x,t} MC_{t+j}^{df}}{S_{t+j} P_{t+j}^{df}}\right] = 0, \tag{121}$$

$$F = \mathrm{E}_t \sum_{j=0}^{\infty} \left(\beta \xi^{df}\right)^j \psi_{z,t+j} \left(\frac{\pi_t^{df}}{\pi_{t+j}^{df}}\right)^{-\frac{\lambda_{x,t}}{\lambda_{x,t-1}}} X_{t+j}^{\#df} \left[\frac{\pi_t^{df}}{\pi_{t+j}^{df}} \widetilde{p}_t - \lambda_{x,t} mc_{t+j}^{df}\right] = 0, \tag{122}$$

where $\psi_{t+j}^{df} \equiv \upsilon_{t+j} P_{t+j}^{df}$, and $X_{t+j}^{\#df} = \frac{X_{t+j}^{df}}{z^{*}{}_{t+j}}.\psi_{z,t} \equiv \upsilon_t P_t^{df} z_t^{*}$, $\widetilde{p}_t^{df} = \frac{P_{new,t}^{df}}{P_t^{df}}$, and $mc_t^{df} = \frac{MC_t^{df}}{S_t P_t^{df}}$.

2.3.1 Log-linearizing the export price equation

A first order Taylor approximation around the steady state is obtained by totally differentiating the export price equation (122) and evaluating it in steady-state. A hat denotes the log deviation from steady state (i.e., $x_t = \frac{dx_t}{x} = \ln X_t - \ln X$). Note that in steady-state; $X_{t+j}^{\#df}$ and $\psi_{z,t+j}^{df}$ are constants, $X^{\#df}$ and ψ_z^{df},respectively, $\widetilde{p}^{df} = \lambda_x mc^{df} = 1, \pi_t = \overline{\pi}$, and $X_{t,j} = \frac{\overline{\pi}}{\overline{\pi}} = 1$.

i.e.,

Totally differentiating (noting that the term within brackets is zero in steady state):

$$\begin{aligned} 0 &= \frac{1}{(1-\beta\xi^{df})}\psi_z^{df} X^{\#df} \widetilde{p}^{df} \frac{d\widetilde{p}_t^{df}}{\widetilde{p}^{df}} + \frac{\beta\xi^{df}}{(1-\beta\xi^{df})}\psi_z^{df} X^{\#df} \left(\frac{1}{\overline{\pi}^{df}}\right) \overline{\pi}^{df} \frac{d\pi_t^{df}}{\overline{\pi}^{df}} \\ &\quad -\sum_{j=1}^{\infty} \left(\beta\xi^{df}\right)^j \psi_z^{df} X^{\#df} \left(\frac{1}{\overline{\pi}^{df}}\right) \overline{\pi}^{df} \frac{d\pi_{t+j}^{df}}{\overline{\pi}^{df}} - \sum_{j=0}^{\infty} \left(\beta\xi^{df}\right)^j \psi_z^{df} X^{\#df} \lambda_x mc^{df} \frac{dmc_{t+j}^{df}}{mc^{df}} \\ &\quad -\sum_{j=0}^{\infty} \left(\beta\xi^{df}\right)^j \psi_z^{df} X^{\#df} mc^{df} \lambda_x \frac{d\lambda_{x,t+j}}{\lambda_x}. \end{aligned} \tag{123}$$

Taking into account that $\lambda_x mc^{df} = \widetilde{p}^{df} = 1$ and rearranging, the Taylor expansion is, consequently;

$$\begin{aligned} 0 &= \frac{\psi_z^{df} X^{\#df}}{(1-\beta\xi^{df})}\widehat{\widetilde{p}}_t^{df} + \frac{\beta\xi^{df}\psi_z^{df} X^{\#df}}{(1-\beta\xi^{df})}\widehat{\pi}_t^{df} - \sum_{j=1}^{\infty} \left(\beta\xi^{df}\right)^j \psi_z^{df} X^{\#df} \widehat{\pi}_{t+j}^{df} \\ &\quad -\sum_{j=0}^{\infty} \left(\beta\xi^{df}\right)^j \psi_z^{df} X^{\#df} \left(\widehat{mc}_{t+j}^{df} + \widehat{\lambda}_{x,t+j}\right). \end{aligned} \tag{124}$$

Solving for $\widehat{\widetilde{p}}_t^{df}$

$$\widehat{\widetilde{p}}_t^{df} = -\beta\xi^{df}\widehat{\pi}_t^{df} + (1-\beta\xi^{df})\sum_{j=1}^{\infty} \left(\beta\xi^{df}\right)^j \widehat{\pi}_{t+j}^{df} + (1-\beta\xi^{df})\sum_{j=0}^{\infty} \left(\beta\xi^{df}\right)^j \left(\widehat{mc}_{t+j}^{df} + \widehat{\lambda}_{x,t+j}\right). \tag{125}$$

From the aggregate price index follows that the average price in period t is:

$$P_t^{df} = \left[\xi^{df}\left(P_{t-1}^{df}\pi_{t-1}^{df}\right)^{\frac{1}{1-\lambda_{x,t}}} + (1-\xi^{df})\left(P_{new,t}^{df}\right)^{\frac{1}{1-\lambda_{x,t}}}\right]^{1-\lambda_{x,t}}. \tag{126}$$

Dividing:

$$1 = \left[\xi^{df}\left(\frac{\pi_{t-1}^{df}}{\pi_t^{df}}\right)^{\frac{1}{1-\lambda_{x,t}}} + (1-\xi^{df})\left(\frac{P_{new,t}^{df}}{P_t^{df}}\right)^{\frac{1}{1-\lambda_{x,t}}}\right]^{1-\lambda_{x,t}}. \tag{127}$$

Linearize equation (127). Start by taking both sides to the power of $\frac{1}{1-\lambda_{f,t}}$:

$$D = \left[\xi^{df}\left(\frac{\pi_{t-1}^{df}}{\pi_t^{df}}\right)^{\frac{1}{1-\lambda_{x,t}}} + (1-\xi^{df})\left(\frac{P_{new,t}^{df}}{P_t^{df}}\right)^{\frac{1}{1-\lambda_{x,t}}}\right] = 1. \tag{128}$$

Differentiating with respect to $\lambda_{x,t}$ we require the derivative with respect to a power. Note that

$$f(\lambda_{x,t}) = x^{\frac{1}{1-\lambda_{x,t}}} = e^{\ln\left(x^{\frac{1}{1-\lambda_{x,t}}}\right)} = e^{\frac{1}{1-\lambda_{x,t}}\ln x}.$$

Then, differentiating this expression with respect to $\lambda_{x,t}$:

$$\begin{aligned}\frac{\partial f}{\partial \lambda_{x,t}} &= e^{\frac{1}{1-\lambda_{x,t}}\ln x}(\ln x)\left(\frac{-1}{(1-\lambda_{x,t})^2}(-1)\right) \\ &= (\ln x)\, x^{\frac{1}{1-\lambda_{x,t}}}\left(\frac{1}{(1-\lambda_{x,t})^2}\right).\end{aligned}$$

It is important to note, that x is defined as; $x = \tilde{p}^{df}$ and $x = \overline{\pi}/\overline{\pi}$ in equation (128). In each case, $x = 1$ in steady state. Note also that $\ln(1) = 0$, so $\hat{\lambda}_{x,t}$ does not appear in the linear expansion of (128). Totally differentiating (128) and evaluating in steady state:

$$dD = \frac{(1-\xi^{df})}{(1-\lambda_x)}\widetilde{p}^{df}\frac{d\widetilde{p}_t^{df}}{\widetilde{p}^{df}} + \frac{\xi^{df}}{(1-\lambda_x)}\frac{1}{\overline{\pi}^{df}}\overline{\pi}^{df}\frac{d\pi_{t-1}^{df}}{\overline{\pi}^{df}} - \frac{\xi^{df}}{(1-\lambda_x)}\frac{1}{\overline{\pi}^{df}}\overline{\pi}^{df}\frac{d\pi_t^{df}}{\overline{\pi}^{df}} = 0.$$

Solving for $\widetilde{\widehat{p}}_t^{df}$:

$$\widetilde{\widehat{p}}_t^{df} = \frac{\xi^{df}}{(1-\xi^{df})}\left(\widehat{\pi}_t^{df} - \widehat{\pi}_{t-1}^{df}\right) = 0. \tag{129}$$

Inserting equation (129) in equation (125):

$$\frac{\xi^{df}}{(1-\xi^{df})}\left(\widehat{\pi}_t^{df} - \widehat{\pi}_{t-1}^{df}\right) = -\beta\xi^{df}\widehat{\pi}_t^{df} + (1-\beta\xi^{df})\sum_{j=1}^{\infty}\left(\beta\xi^{df}\right)^j \widehat{\pi}_{t+j}^{df}$$
$$+(1-\beta\xi^{df})\sum_{j=0}^{\infty}\left(\beta\xi^d f\right)^j\left(\widehat{mc}_{t+j}^{df} + \widehat{\lambda}_{x,t+j}\right).$$

Simplifying:

$$\left[1+(1-\xi^{df})\beta\right]\pi_t^{df} - \widehat{\pi}_{t-1}^{df} = \frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}}\sum_{j=1}^{\infty}\left(\beta\xi^{df}\right)^j \widehat{\pi}_{t+j}^{df} \tag{130}$$
$$+\frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}}\sum_{j=0}^{\infty}\left(\beta\xi^{df}\right)^j\left(\widehat{mc}_{t+j}^{df} + \widehat{\lambda}_{x,t+j}\right).$$

Lead this expression forward by one period and multiply by $\beta\xi^{df}$:

$$\beta\xi^{df}\left[1+(1-\xi^{df})\beta\right]\widehat{\pi}_{t+1}^{df} - \beta\xi^{df}\widehat{\pi}_t^{df} = \frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}}\sum_{j=2}^{\infty}\left(\beta\xi^{df}\right)^j \widehat{\pi}_{t+j}^{df} \tag{131}$$
$$+\frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}}\sum_{j=1}^{\infty}\left(\beta\xi^{df}\right)^j\left(\widehat{mc}_{t+j}^{df} + \widehat{\lambda}_{x,t+j}\right).$$

Subtract equation (131) from equation (130), to obtain:

$$\left[1+(1-\xi^{df})\beta+\beta\xi^{df}\right]\widehat{\pi}_t^{df} - \beta\xi^{df}\left[1+(1-\xi^{df})\beta\right]\widehat{\pi}_{t+1}^{df} - \widehat{\pi}_{t-1}^{df}$$
$$= \frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}}\left(\beta\xi^{df}\right)\widehat{\pi}_{t+1}^{df} + \frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}}(\beta\xi)^0\left(\widehat{mc}_t^{df} + \widehat{\lambda}_{x,t}\right) \tag{132}$$

Combining terms and simplifying:

$$\left[1+(1-\xi^{df})\beta+\beta\xi^{df}\right]\widehat{\pi}_t^{df} - \widehat{\pi}_{t-1}^{df}$$
$$= \beta\xi^{df}\left[1+(1-\xi^{df})\beta\right]\widehat{\pi}_{t+1}^{df} + (1-\xi^{df})(1-\beta\xi^{df})\beta\widehat{\pi}_{t+1}^{df} + \frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}}\left(\widehat{mc}_t^{df} + \widehat{\lambda}_{x,t}\right) \tag{133}$$

or,

$$\widehat{\pi}_t^{df} = \frac{\beta}{1+\beta}\mathrm{E}_t\widehat{\pi}_{t+1}^{df} + \frac{1}{1+\beta}\widehat{\pi}_{t-1}^{df} + \frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}(1+\beta)}\left(\widehat{mc}_t^{df} + \widehat{\lambda}_{x,t}\right). \quad (134)$$

Note that the marginal cost is the input price $MC_t^{df} = P_t^d$. Consequently, $mc_t^{df} = \frac{MC_t^{df}}{S_t P_t^{df}} = \frac{P_t}{S_t P_t^{df}}$. Log-linearizing this expression and evaluating in steady state ($mc = \frac{P}{SP^{df}}$):

$$mc^{df}\widehat{mc}_t^{df} = \frac{P}{SP^{df}}\widehat{P}_t - \frac{P}{S}\frac{1}{P^{df}}\widehat{P}_t^{df} - \frac{P}{P^{df}}\frac{1}{S}\widehat{S}_t, \quad (135)$$

$$\widehat{mc}_t^{df} = \left(\widehat{P}_t - \widehat{P}_t^{df} - \widehat{S}_t\right). \quad (136)$$

Using this expression in the log-linearized Phillips curve relation, the export inflation will follow

$$\widehat{\pi}_t^{df} = \frac{\beta}{1+\beta}\mathrm{E}_t\widehat{\pi}_{t+1}^{df} + \frac{1}{1+\beta}\widehat{\pi}_{t-1}^{df} + \frac{(1-\xi^{df})(1-\beta\xi^{df})}{\xi^{df}(1+\beta)}\left(\widehat{P}_t - \widehat{P}_t^{df} - \widehat{S}_t + \widehat{\lambda}_{x,t}\right). \quad (137)$$

Note also that the deviations from the Law of One Price will follow

$$\begin{aligned} \widehat{mc}_t^{df} &= \widehat{P}_t - \widehat{P}_t^{df} - \widehat{S}_t \\ &= \widehat{mc}_{t-1}^{df} + \widehat{\pi}_t - \widehat{\pi}_t^{df} - \Delta\widehat{S}_t \end{aligned} \quad (138)$$

3 Log-linearization of foc:s for u, q, b^*, and the UIP

3.1 Log-linearizing the capital utilization equation

The first order condition for capital utilization follows

$$F = \psi_{z,t}\left[(1-\tau_t^k)r_t^k - a'(u_t)\right] = 0. \tag{139}$$

Note that in steady state the following holds; $\psi_{z,t}$ is a constant ψ_z, $u = 1$, $a(u) = a(1) = 1$, $(1-\tau^k)r^k = a'(u)$. The Taylor expansion can be written as

$$\frac{dF}{F} = \psi_z F_\psi \frac{d\psi_{z,t}}{\psi_z} + \tau^k F_\tau \frac{d\tau_t^k}{\tau^k} + r^k F_r \frac{dr_t^k}{r^k} + uF_u \frac{du_t}{u} = 0. \tag{140}$$

Differentiating equation (139) with respect to the variables $\psi_{z,t}$, τ_t^k, r_t^k and u_t, and evaluating these derivatives in steady state:

$$\frac{\partial F}{\partial \psi_{z,t}} = 0.$$

$$\frac{\partial F}{\partial \tau_t^k} = -\psi_z r^k.$$

$$\frac{\partial F}{\partial r_t^k} = \psi_z(1-\tau^k).$$

$$\frac{\partial F}{\partial u_t} = -\psi_z a''(u).$$

Collecting terms and inserting in equation (140) implies that the Taylor expansion follows:

$$0 = \tau^k(-\psi_z r^k)\widehat{\tau}_t^k + r^k \psi_z(1-\tau^k)\widehat{r}_t^k + u[-\psi_z a''(u)]\widehat{u}_t. \tag{141}$$

Dividing by $\psi_z r^k(1-\tau^k)$

$$0 = -\frac{\tau^k}{(1-\tau^k)}\widehat{\tau}_t^k + \widehat{r}_t^k - \frac{a''(u)}{(1-\tau^k)r^k}\widehat{u}_t.$$

Let $\sigma_a = \frac{a''}{a'}$

$$0 = \widehat{r}_t^k - \sigma_a \widehat{u}_t - \frac{\tau^k}{(1-\tau^k)} \widehat{\tau}_t^k. \tag{142}$$

Solving for $\widehat{u}_t$:

$$\widehat{u}_t = \frac{1}{\sigma_a} \widehat{r}_t^k - \frac{1}{\sigma_a} \frac{\tau^k}{(1-\tau^k)} \widehat{\tau}_t^k. \tag{143}$$

3.2 Log-linearizing the real cash balances equation

The first order condition for the real cash balances is given by

$$\zeta_t^q V'(\frac{Q_t}{z_t P_t}) \frac{1}{z_t} - \left(1 - \tau_t^k\right) \psi_t \left(R_{t-1} - 1\right) = 0. \tag{144}$$

Note that the V(.) function is given by $V(.) = A_q \frac{(.)}{1-\sigma_q}^{1-\sigma_q}$, so this can be written

$$F = \zeta_t^q A_q q_t^{-\sigma_q} - \left(1 - \tau_t^k\right) \psi_{z,t} \left(R_{t-1} - 1\right) = 0. \tag{145}$$

Differentiating equation (145) with respect to the variables ζ_t^q, q_t, τ_t^k, $\psi_{z,t}$, and R_{t-1}, and evaluating in steady state

$$\frac{\partial F}{\partial \zeta_t^q} = A_q q^{-\sigma_q}. \tag{146}$$

$$\frac{\partial F}{\partial q_t} = -\sigma_q \zeta^q A_q q^{-\sigma_q - 1}. \tag{147}$$

$$\frac{\partial F}{\partial \tau_t^k} = \psi_z \left(R - 1\right). \tag{148}$$

$$\frac{\partial F}{\partial \psi_{z,t}} = -\left(1 - \tau^k\right) \left(R - 1\right). \tag{149}$$

$$\frac{\partial F}{\partial R_{t-1}} = -\left(1 - \tau^k\right) \psi_z. \tag{150}$$

After collecting terms the Taylor expansion follows:

$$\begin{aligned} 0 &= \zeta^q A_q q^{-\sigma_q} \widehat{\zeta}_t^q - q\sigma_q \zeta^q A_q q^{-\sigma_q - 1} \widehat{q}_t + \tau^k \psi_z (R-1) \widehat{\tau}_t^k \\ & -\psi_z \left(1-\tau^k\right)(R-1) \widehat{\psi}_{z,t} - R\left(1-\tau^k\right)\psi_z \widehat{R}_{t-1}. \end{aligned} \tag{151}$$

Note that the following holds in steady state (see Jesper's notes; equation 9.27 (q_ss))

$$A_q q^{-\sigma_q} = \left(1-\tau^k\right)\psi_z (R-1).$$

Inserting this implies

$$\begin{aligned} 0 &= \zeta^q \left(1-\tau^k\right)\psi_z (R-1) \widehat{\zeta}_t^q - \sigma_q \zeta^q \left(1-\tau^k\right)\psi_z (R-1) \widehat{q}_t \\ & +\tau^k \psi_z (R-1) \widehat{\tau}_t^k - \psi_z \left(1-\tau^k\right)(R-1) \widehat{\psi}_{z,t} - R\left(1-\tau^k\right)\psi_z \widehat{R}_{t-1} \end{aligned} \tag{152}$$

Dividing by $\left(1-\tau^k\right)\psi_z (R-1)$

$$0 = \zeta^q \widehat{\zeta}_t^q - \sigma_q \zeta^q \widehat{q}_t + \frac{\tau^k}{1-\tau^k}\widehat{\tau}_t^k - \widehat{\psi}_{z,t} - \frac{R}{R-1}\widehat{R}_{t-1}. \tag{153}$$

Solving for $\widehat{q}_t$,and noting that ζ^q=1 in steady state:

$$\widehat{q}_t = \frac{1}{\sigma_q}\left[\widehat{\zeta}_t^q + \frac{\tau^k}{1-\tau^k}\widehat{\tau}_t^k - \widehat{\psi}_{z,t} - \frac{R}{R-1}\widehat{R}_{t-1}\right]. \tag{154}$$

3.3 Log-linearizing the foreign bonds equation

The first order condition for the holding of foreign bonds follow

$$F = -\psi_{z,t} S_t + \beta E_t \left[\frac{\psi_{z,t+1}}{\mu_{z,t+1}} \frac{S_{t+1}}{\pi_{t+1}} \left(R_t^* \Phi(a_t, \widetilde{\phi}_t) - \tau_{t+1}^k \left(R_t^* \Phi(a_t, \widetilde{\phi}_t) - 1\right)\right)\right] = 0. \tag{155}$$

Differentiate this with respect to the variables $\psi_{z,t}$, $\psi_{z,t+1}$, S_t, $\mu_{z,t+1}$, a_t, $\widetilde{\phi}_t$, S_{t+1}, π_{t+1}, R_t^*, τ_{t+1}^k, and evaluate the derivatives in steady state. Note that the steady state is defined as $S=1$, $\Phi(a,\widetilde{\phi})=\Phi(0,0)=1$ (where $a=\frac{A}{z},\widetilde{\phi}=0$), $R=R^{*}$, and $\left(R^{*}-\tau^{k}\left(R^{*}-1\right)\right)=\frac{\mu_{z}\overline{\pi}}{\beta}$. Note also that we assume $\Phi(a,\widetilde{\phi})=e^{(-\widetilde{\phi}_{a}a,+\widetilde{\phi})}$.This implies that $\Phi_{a}(a,0)=-\widetilde{\phi}_{a}$, and $\Phi_{\phi}(a,0)=1$. The Taylor expansion follows

$$\begin{aligned} 0=\tfrac{dF}{F} &= \psi_{z}F_{\psi}\tfrac{d\psi_{z,t}}{\psi_{z}}+\psi_{z}F_{\psi}\tfrac{d\psi_{z,t+1}}{\psi_{z}}+sF_{s_{t}}\tfrac{ds_{t}}{s}+sF_{s_{t+1}}\tfrac{ds_{t+1}}{s}+\mu_{z}F_{\mu}\tfrac{d\mu_{z,t+1}}{\mu_{z}} \\ &+F_{a}da_{t}+F_{\phi}d\widetilde{\phi}_{t}+\pi F_{\pi}\tfrac{d\pi_{t+1}}{\pi}+R^{*}F_{R^{*}}\tfrac{dR_{t}^{*}}{R^{*}}+\tau^{k}F_{\tau}\tfrac{d\tau_{t+1}^{k}}{\tau^{k}}. \end{aligned} \quad (156)$$

$$\begin{aligned} \frac{\partial F}{\partial\psi_{z,t}} &= -S \\ &= -1. \end{aligned}$$

$$\begin{aligned} \frac{\partial F}{\partial\psi_{z,t+1}} &= \beta\frac{1}{\mu_{z,t+1}}\frac{S}{\overline{\pi}}\left(R^{*}\Phi(a,\widetilde{\phi})-\tau^{k}\left(R^{*}\Phi(a,\widetilde{\phi})-1\right)\right) \\ &= 1. \end{aligned}$$

$$\frac{\partial F}{\partial S_{t}}=-\psi_{z}.$$

$$\begin{aligned} \frac{\partial F}{\partial S_{t+1}} &= \beta\frac{\psi_{z}}{\mu_{z}}\frac{1}{\pi}\left(R^{*}\Phi(a,\widetilde{\phi})-\tau^{k}\left(R^{*}\Phi(a,\widetilde{\phi})-1\right)\right) \\ &= \psi_{z}. \end{aligned}$$

$$\begin{aligned} \frac{\partial F}{\partial\mu_{z,t+1}} &= \beta\psi_{z}\frac{S}{\overline{\pi}}\left(R^{*}\Phi(a,\widetilde{\phi})-\tau^{k}\left(R^{*}\Phi(a,\widetilde{\phi})-1\right)\right)\frac{-1}{\left(\mu_{z}\right)^{2}} \\ &= \frac{\beta\psi_{z}}{\overline{\pi}}\frac{\mu_{z}\overline{\pi}}{\beta}\frac{-1}{\left(\mu_{z}\right)^{2}} \\ &= \frac{-\psi_{z}}{\mu_{z}}. \end{aligned}$$

$$\begin{aligned} \frac{\partial F}{\partial a_{t}} &= \beta\frac{\psi_{z}}{\mu_{z}}\frac{S}{\overline{\pi}}\left(R^{*}-\tau^{k}R^{*}\right)\Phi_{a}(a,\widetilde{\phi}) \\ &= \beta\frac{\psi_{z}}{\mu_{z}}\frac{S}{\overline{\pi}}R^{*}\left(1-\tau^{k}\right)\Phi_{a}(a,\widetilde{\phi}) \\ &= -\frac{\beta\psi_{z}\left(1-\tau^{k}\right)}{\mu_{z}\overline{\pi}}R\widetilde{\phi}_{a}. \end{aligned}$$

$$
\begin{aligned}
\frac{\partial F}{\partial \widetilde{\phi}_t} &= \beta \frac{\psi_z}{\mu_z} \frac{S}{\overline{\pi}} \left(R^* - \tau^k R^* \right) \Phi_\phi(a, \widetilde{\phi}) \\
&= \beta \frac{\psi_z}{\mu_z} \frac{S}{\overline{\pi}} R^* \left(1 - \tau^k \right) \Phi_\phi(a, \widetilde{\phi}) \\
&= \frac{\beta \psi_z \left(1 - \tau^k\right)}{\mu_z \overline{\pi}} R.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial \pi_{t+1}} &= \beta \frac{\psi_z}{\mu_z} S \left(R^* \Phi(a, \widetilde{\phi}) - \tau^k \left(R^* \Phi(a, \widetilde{\phi}) - 1 \right) \right) \frac{-1}{(\overline{\pi})^2} \\
&= \frac{\beta \psi_z}{\mu_z} \frac{\mu_z \overline{\pi}}{\beta} \frac{-1}{(\overline{\pi})^2} \\
&= \frac{-\psi_z}{\overline{\pi}}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial R^*_t} &= \beta \frac{\psi_z}{\mu_z} \frac{S}{\overline{\pi}} \Phi(a, \widetilde{\phi}) \left(1 - \tau^k\right) \\
&= \frac{\beta \psi_z \left(1 - \tau^k\right)}{\mu_z \overline{\pi}}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial \tau^k_{t+1}} &= \beta \frac{\psi_z}{\mu_z} \frac{S}{\overline{\pi}} \left(- \left(R^* \Phi(a, \widetilde{\phi}) - 1 \right) \right) \\
&= \frac{-\beta \psi_z}{\mu_z \overline{\pi}} \left(R - 1 \right).
\end{aligned}
$$

Note that the steady state interest rate is $R = \frac{\pi \mu_z - \tau^k \beta}{(1-\tau^k)\beta}$ which implies that the latter expression can be rewritten as

$$
\frac{\partial F}{\partial \tau^k_{t+1}} = \frac{-\psi_z}{\mu_z \overline{\pi}} \frac{(\overline{\pi} \mu_z - \beta)}{(1 - \tau^k)}.
$$

Collecting terms and inserting in equation (156):

$$
\begin{aligned}
0 \quad = \quad & -\psi_z \widehat{\psi}_{z,t} + \psi_z \widehat{\psi}_{z,t+1} - S \psi_z \widehat{S}_t + S \psi_z \widehat{S}_{t+1} + \mu_z \frac{-\psi_z}{\mu_z} \widehat{\mu}_{z,t+1} - \frac{\beta \psi_z \left(1 - \tau^k\right)}{\mu_z \overline{\pi}} R \widetilde{\phi} da_t \\
& + \frac{\beta \psi_z \left(1 - \tau^k\right)}{\mu_z \overline{\pi}} R d\widetilde{\phi}_t + \pi \frac{-\psi_z}{\overline{\pi}} \widehat{\pi}_{t+1} + R^* \frac{\beta \psi_z \left(1 - \tau^k\right)}{\mu_z \overline{\pi}} \widehat{R}^*_t - \tau^k \frac{\psi_z}{\mu_z \overline{\pi}} \frac{(\overline{\pi} \mu_z - \beta)}{(1 - \tau^k)} \widehat{\tau}^k_{t+1}
\end{aligned}
\tag{157}
$$

Using the steady state conditions and dividing by ψ_z, this simplifies to:

$$
\begin{aligned}
0 \quad = \quad & \left(\widehat{\psi}_{z,t+1} - \widehat{\psi}_{z,t} \right) + \left(\widehat{S}_{t+1} - \widehat{S}_t \right) - \widehat{\mu}_{z,t+1} - \frac{\beta \left(1 - \tau^k\right)}{\mu_z \overline{\pi}} \frac{\overline{\pi} \mu_z - \tau^k \beta}{(1 - \tau^k) \beta} \left[\widetilde{\phi} da_t - d\widetilde{\phi}_t \right] - \widehat{\pi}_{t+1} \\
& + \frac{(\overline{\pi} \mu_z - \tau^k \beta)}{(1 - \tau^k) \beta} \frac{\beta \left(1 - \tau^k\right)}{\mu_z \overline{\pi}} \widehat{R}^*_t - \tau^k \frac{1}{\mu_z \overline{\pi}} \frac{(\overline{\pi} \mu_z - \beta)}{(1 - \tau^k)} \widehat{\tau}^k_{t+1},
\end{aligned}
\tag{158}
$$

$$0 = \left(\widehat{\psi}_{z,t+1} - \widehat{\psi}_{z,t}\right) + \left(\widehat{S}_{t+1} - \widehat{S}_t\right) - \widehat{\mu}_{z,t+1} - \frac{(\overline{\pi}\mu_z - \tau^k\beta)}{\mu_z\overline{\pi}}\left[\widetilde{\phi}da_t - d\widetilde{\phi}_t\right] - \widehat{\pi}_{t+1}$$
$$+\frac{(\overline{\pi}\mu_z - \tau^k\beta)}{\mu_z\overline{\pi}}\widehat{R}_t^* - \frac{\tau^k}{(1-\tau^k)}\frac{(\overline{\pi}\mu_z - \beta)}{\mu_z\overline{\pi}}\widehat{\tau}_{t+1}^k. \quad (159)$$

3.4 The UIP condition

The first order conditions for m_{t+1} and b_{t+1}^* are respectively:

$$-\psi_{z,t} + \beta E_t\left[\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\frac{1}{\pi_{t+1}}\left(R_t - \tau_{t+1}^k\left(R_t - 1\right)\right)\right] = 0. \quad (160)$$

$$-\psi_{z,t} + \frac{\beta}{S_t}E_t\left[\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\frac{S_{t+1}}{\pi_{t+1}}\left(R_t^*\Phi(a_t,\widetilde{\phi}_t) - \tau_{t+1}^k\left(R_t^*\Phi(a_t,\widetilde{\phi}_t) - 1\right)\right)\right] = 0. \quad (161)$$

Totally differentiating the first expression (the foc for m_{t+1}) implies

$$0 = -d\psi_{z,t} + \beta E_t[\frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(R - \tau^k\left(R-1\right)\right)\widehat{\psi}_{z,t+1} - \frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(R - \tau^k\left(R-1\right)\right)\widehat{\pi}_{t+1}$$
$$+\frac{\psi_z}{\mu_z}\frac{1}{\pi}R(1-\tau^k)\widehat{R}_t - \frac{\psi_z}{\mu_z}\frac{1}{\pi}(R-1)\tau^k\widehat{\tau}_{t+1}^k]. \quad (162)$$

Divide by ψ_z

$$0 = -\widehat{\psi}_{z,t} + \beta E_t[\frac{1}{\mu_z}\frac{1}{\pi}\left(R - \tau^k\left(R-1\right)\right)\widehat{\psi}_{z,t+1} - \frac{1}{\mu_z}\frac{1}{\pi}\left(R - \tau^k\left(R-1\right)\right)\widehat{\pi}_{t+1}$$
$$+\frac{1}{\mu_z}\frac{1}{\pi}R(1-\tau^k)\widehat{R}_t - \frac{1}{\mu_z}\frac{1}{\pi}(R-1)\tau^k\widehat{\tau}_{t+1}^k]. \quad (163)$$

Totally differentiating the foc for foreign bond holdings;

$$0 = -d\psi_{z,t} - \frac{1}{S}\beta E_t\left[\frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(R^* - \tau^k\left(R^*-1\right)\right)\right]\widehat{S}_t$$
$$+\beta E_t[\frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(R^* - \tau^k\left(R^*-1\right)\right)\widehat{S}_{t+1} - \frac{\psi_z}{\mu_z}\frac{1}{\pi}R^*(1-\tau^k)\widetilde{\phi}da_t$$
$$+\frac{\psi_z}{\mu_z}\frac{1}{\pi}R^*(1-\tau^k)d\widetilde{\phi}_t + \frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(R^* - \tau^k\left(R^*-1\right)\right)\widehat{\psi}_{z,t+1}$$
$$-\frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(R^* - \tau^k\left(R^*-1\right)\right)\widehat{\pi}_{t+1} + \frac{\psi_z}{\mu_z}\frac{1}{\pi}R^*(1-\tau^k)\widehat{R}_t^* - \frac{\psi_z}{\mu_z}\frac{1}{\pi}(R^*-1)\tau^k\widehat{\tau}_{t+1}^k \quad (164)$$

Divide by ψ_z, and using that $R = R^*$;

$$\begin{aligned}
0 &= -\widehat{\psi}_{z,t} - \beta E_t \left[\frac{1}{\mu_z} \frac{1}{\pi} \left(R - \tau^k (R-1) \right) \right] \widehat{S}_t \\
&+\beta E_t [\frac{1}{\mu_z} \frac{1}{\pi} \left(R - \tau^k (R-1) \right) \widehat{S}_{t+1} - \frac{1}{\mu_z} \frac{1}{\pi} R^* (1-\tau^k) \left[\widetilde{\phi} da_t - d\widetilde{\phi}_t \right] \\
&+ \frac{1}{\mu_z} \frac{1}{\pi} \left(R - \tau^k (R-1) \right) \widehat{\psi}_{z,t+1} - \frac{1}{\mu_z} \frac{1}{\pi} \left(R - \tau^k (R-1) \right) \widehat{\pi}_{t+1} \\
&+ \frac{1}{\mu_z} \frac{1}{\pi} R (1-\tau^k) \widehat{R}^*_t - \frac{1}{\mu_z} \frac{1}{\pi} (R-1) \tau^k \widehat{\tau}^k_{t+1}]
\end{aligned} \tag{165}$$

Equating equations (163) and (165) implies

$$\begin{aligned}
\beta E_t \left[\frac{1}{\mu_z} \frac{1}{\pi} R(1-\tau^k) \widehat{R}_t \right] &= -\beta E_t \left[\frac{1}{\mu_z} \frac{1}{\pi} \left(R - \tau^k (R-1) \right) \right] \widehat{S}_t \\
&- \beta E_t \left[\frac{1}{\mu_z} \frac{1}{\pi} R(1-\tau^k) \left(\widetilde{\phi} da_t - d\widetilde{\phi}_t \right) \right] \\
&+ \beta E_t \left[\frac{1}{\mu_z} \frac{1}{\pi} \left(R - \tau^k (R-1) \right) \widehat{S}_{t+1} + \frac{1}{\mu_z} \frac{1}{\pi} R(1-\tau^k) \widehat{R}^*_t \right]
\end{aligned} \tag{166}$$

By noting that the following holds in steady state $S = 1$, $\Phi(a, \widetilde{\phi}) = \Phi(0,0) = 1, R = R^*$, $R = \frac{\pi\mu_z - \tau^k \beta}{(1-\tau^k)\beta}$, and $\left(R^* - \tau^k (R^* - 1) \right) = \frac{\mu_z \overline{\pi}}{\beta}$. This is equivalent to

$$\begin{aligned}
\widehat{R}_t &= \frac{\mu_z \overline{\pi}}{R(1-\tau^k)} [-\frac{1}{\beta} \widehat{S}_t - \frac{1}{\mu_z} \frac{1}{\overline{\pi}} R(1-\tau^k) \left(\widetilde{\phi} da_t - d\widetilde{\phi}_t \right) \\
&+ \frac{1}{\beta} E_t \widehat{S}_{t+1} + \frac{1}{\mu_z} \frac{1}{\overline{\pi}} R(1-\tau^k) \widehat{R}^*_t] \\
&= \frac{(1-\tau^k)\beta}{\pi\mu_z - \tau^k \beta} \frac{\mu_z \overline{\pi}}{(1-\tau^k)} \left[-\frac{1}{\beta} \widehat{S}_t + \frac{1}{\beta} E_t \widehat{S}_{t+1} \right] - \widetilde{\phi} da_t + d\widetilde{\phi}_t + \widehat{R}^*_t.
\end{aligned} \tag{167}$$

This implies that the UIP-condition follows:

$$\widehat{R}_t - \widehat{R}^*_t = \frac{\mu_z \pi}{\pi \mu_z - \tau^k \beta} \left[E_t \widehat{S}_{t+1} - \widehat{S}_t \right] - \widetilde{\phi} da_t + d\widetilde{\phi}_t. \tag{168}$$

$$0 = -\left(\widehat{R}_t - \widehat{R}^*_t \right) + \frac{\mu_z \pi}{\pi \mu_z - \tau^k \beta} \left[E_t \widehat{S}_{t+1} - \widehat{S}_t \right] - \widetilde{\phi} da_t + d\widetilde{\phi}_t \tag{169}$$

4 Profits

4.1 Domestic firms

Profits for the (aggregate) domestic firms are given by:

$$\Pi_t^d = P_t^d \left(C_t^d + I_t^d + G_t\right) + P_t^{df} S_t \left(C_t^{df} + I_t^{df}\right) - MC_t(C_t^d + I_t^d + G_t + C_t^{df} + I_t^{df}) - MC_t z_t \phi, \tag{170}$$

$$\Pi_t^d = P_t^d \left(C_t^d + I_t^d + G_t\right) + P_t^d \left(C_t^{df} + I_t^{df}\right) - MC_t(C_t^d + I_t^d + G_t + C_t^{df} + I_t^{df}) - MC_t z_t \phi. \tag{171}$$

Stationarize equation (171)

$$\frac{\Pi_t^d}{P_t^d z_t} = \frac{P_t^d \left(C_t^d + I_t^d + G_t\right)}{P_t^d z_t} + \frac{P_t^d \left(C_t^{df} + I_t^{df}\right)}{P_t^d z_t} - \frac{MC_t(C_t^d + I_t^d + G_t + C_t^{df} + I_t^{df})}{P_t^d z_t} - \frac{MC_t z_t \phi}{P_t^d z_t}. \tag{172}$$

Let lower case letters denote that a variable has been stationarized $x_t = X_t / z_t$:

$$\overline{\Pi}_t^d = c_t^d + i_t^d + g_t + c_t^{df} + i_t^{df} - mc_t(c_t^d + i_t^d + g_t + c_t^{df} + i_t^{df}) - mc_t \phi, \tag{173}$$

where $mc_t = \frac{MC_t}{P_t^d}$, and $\overline{\Pi}_t^d = \frac{\Pi_t^d}{P_t^d z_t}$. Differentiate this expression with respect to the variables c_t^d, i_t^d, g_t, c_t^{df}, i_t^{df}, mc_t, evaluating in steady state

$$\frac{\partial \overline{\Pi}_t^d}{\partial c_t^d} = 1 - mc.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial i_t^d} = 1 - mc.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial g_t} = 1 - mc.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial c_t^{df}} = 1 - mc.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial i_t^{df}} = 1 - mc.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial mc_t} = -(c^d + i^d + g + c^{df} + i^{df} + \phi).$$

The Taylor expansion follows

$$\begin{aligned}
\widehat{\overline{\Pi}}_t^d &= c^d(1-mc)\widehat{c}_t^d + i^d(1-mc)\widehat{i}_t^d + g(1-mc)\widehat{g}_t + c^{df}(1-mc)\widehat{c}_t^{df} \\
&\quad + i^{df}(1-mc)\widehat{i}_t^{df} - mc(c^d + i^d + g + c^{df} + i^{df} + \phi)\widehat{mc}_t.
\end{aligned}$$

In steady state we know that $\widetilde{p} = \lambda_f mc = 1$, and $S = 1$. This implies that

$$\begin{aligned}
\widehat{\overline{\Pi}}_t^d &= \left(\frac{\lambda_f - 1}{\lambda_f}\right)\left[c^d\widehat{c}_t^d + i^d\widehat{i}_t^d + g\widehat{g}_t + c^{df}\widehat{c}_t^{df} + i^{df}\widehat{i}_t^{df}\right] \\
&\quad - \frac{1}{\lambda_f}(c^d + i^d + g + c^{df} + i^{df} + \phi)\widehat{mc}_t. \qquad (174)
\end{aligned}$$

Note that the profit function also can be written as

$$\Pi_t^d = P_t^d Y_t - MC_t Y_t - MC_t z_t \phi. \qquad (175)$$

Stationarize this expression:

$$\frac{\Pi_t^d}{P_t^d z_t} = \frac{P_t^d Y_t}{P_t^d z_t} - \frac{MC_t Y_t}{P_t^d z_t} - \frac{MC_t z_t \phi}{P_t^d z_t},$$

$$\overline{\Pi}_t^d = y_t - mc_t y_t - mc_t \phi.$$

Differentiate this with respect to y_t, and mc_t:

$$\frac{\partial \overline{\Pi}_t^d}{\partial y_t} = (1 - mc).$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial mc_t} = -(y + \phi).$$

The Taylor expansion of the profit function in equation (175) then follows

$$\begin{aligned}
\widehat{\overline{\Pi}}_t^d &= y(1-mc)\widehat{y}_t - mc(y+\phi)\widehat{mc}_t \\
&= y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\widehat{y}_t - \frac{1}{\lambda_f}(y+\phi)\widehat{mc}_t
\end{aligned}$$

The profit can also be written in terms of factor prices:

$$\Pi_t^d = P_t^d Y_t - R_t^k K_t - W_t R_t^f H_t - MC_t z_t \phi. \tag{176}$$

Stationarize this expression

$$\frac{\Pi_t^d}{P_t^d z_t} = \frac{P_t^d Y_t}{P_t^d z_t} - \frac{R_t^k K_t z_{t-1}}{P_t^d z_t z_{t-1}} - \frac{W_t R_t^f H_t}{P_t^d z_t} - \frac{MC_t z_t \phi}{P_t^d z_t}.$$

$$\overline{\Pi}_t^d = y_t - r_t^k k_t \frac{1}{\mu_{z,t}} - \overline{w}_t R_t^f H_t - mc_t \phi.$$

Differentiate this expression with respect to the variables y_t, r_t^k, k_t, $\mu_{z,t}$, $\overline{w}_t$, R_t^f, H_t, and mc_t :

$$\frac{\partial \overline{\Pi}_t^d}{\partial y_t} = 1.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial r_t^k} = -k \frac{1}{\mu_z}.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial k_t} = -r^k \frac{1}{\mu_z}.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial \mu_{z,t}} = -r^k k \frac{-1}{(\mu_z)^2}.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial \overline{w}_t} = -R^f H.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial R_t^f} = -\overline{w} H.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial H_t} = -\overline{w} R^f.$$

$$\frac{\partial \overline{\Pi}_t^d}{\partial mc_t} = -\phi.$$

The Taylor expansion follows

$$\begin{aligned} \widehat{\overline{\Pi}}_t^d &= y\widehat{y}_t - r^k k \frac{1}{\mu_z} \widehat{r}_t^k - k r^k \frac{1}{\mu_z} \widehat{k}_t + \mu_z r^k k \frac{1}{(\mu_z)^2} \widehat{\mu}_{z,t} - \overline{w} R^f H \widehat{\overline{w}}_t \\ &\quad - R^f \overline{w} R^f \widehat{R}_t^f - H \overline{w} R^f \widehat{H}_t - mc\phi \widehat{mc}_t \\ &= y\widehat{y}_t - r^k k \frac{1}{\mu_z} \left(\widehat{r}_t^k + \widehat{k}_t - \widehat{\mu}_{z,t} \right) - \overline{w} R^f H \widehat{\overline{w}}_t - R^f \overline{w} R^f \widehat{R}_t^f - H \overline{w} R^f \widehat{H}_t - mc\phi \widehat{mc}_t \end{aligned} \tag{177}$$

4.2 Importing firms

Total profits for the (average) consumption and investment importing firms, are given by:

$$\Pi_t^m = \Pi_t^{m,c} + \Pi_t^{m,i} \tag{178}$$

$$= P_t^{m,c} C_t^m + P_t^{m,i} I_t^m - S_t P_t^* \left(C_t^m + I_t^m \right). \tag{179}$$

Stationarize this

$$\frac{\Pi_t^m}{P_t^d z_t} = \frac{P_t^{m,c} C_t^m}{P_t^d z_t} + \frac{P_t^{m,i} I_t^m}{P_t^d z_t} - \frac{S_t P_t^* \left(C_t^m + I_t^m \right)}{P_t^d z_t}, \tag{180}$$

$$\begin{aligned} \overline{\overline{\Pi}}_t^m &= \gamma_t^{mc,d} c_t^m + \gamma_t^{mi,d} i_t^m - \gamma_t^x \left(c_t^m + i_t^m \right) \\ &= \left(\gamma_t^{mc,d} - \gamma_t^x \right) c_t^m + \left(\gamma_t^{mi,d} - \gamma_t^x \right) i_t^m, \end{aligned}$$

where $\overline{\overline{\Pi}}_t^m = \frac{\Pi_t^m}{P_t^d z_t}$, $\gamma_t^{mc,d} = \frac{P_t^{m,c}}{P_t^d}$, $\gamma_t^{mi,d} = \frac{P_t^{m,i}}{P_t^d}$, and $\gamma_t^x = \frac{S_t P_t^*}{P_t^d}$.

Differentiate this expression with respect to the variables c_t^m, i_t^m, $\gamma_t^{mc,d}$, $\gamma_t^{mi,d}$, γ_t^x, evaluating in steady state:

$$\frac{\partial \overline{\Pi}_t^m}{\partial c_t^m} = \left(\gamma^{mc,d} - \gamma^x \right).$$

$$\frac{\partial \overline{\Pi}_t^m}{\partial i_t^m} = \left(\gamma^{mi,d} - \gamma^x \right).$$

$$\frac{\partial \overline{\Pi}_t^m}{\partial \gamma_t^{mc,d}} = c^m.$$

$$\frac{\partial \overline{\Pi}_t^m}{\partial \gamma_t^{mi,d}} = i^m.$$

$$\frac{\partial \overline{\Pi}_t^m}{\partial \gamma_t^x} = -(c^m + i^m).$$

The Taylor expansion follows

$$\widehat{\overline{\Pi}}_t^m = c^m \left(\gamma^{mc,d} - \gamma^x\right) \widehat{c}_t^m + i^m \left(\gamma^{mi,d} - \gamma^x\right) \widehat{i}_t^m + \gamma^{mc,d} c^m \widehat{\gamma}_t^{mc,d} + \gamma^{mi,d} i^m \widehat{\gamma}_t^{mi,d} - \gamma^x (c^m + i^m) \widehat{\gamma}_t^x. \tag{181}$$

This can be rewritten, using the relative prices in different and more comprehensive notation

$$\begin{aligned} \widehat{\overline{\Pi}}_t^m &= c^m \left(\frac{P^{m,c}}{SP^*}\right) \widehat{c}_t^m + i^m \left(\frac{P^{m,i}}{SP^*}\right) \widehat{i}_t^m + \left(\frac{P^{m,c}}{P^d}\right) c^m \left(\widehat{P}_t^{m,c} - \widehat{P}_t^d\right) \\ &+ \left(\frac{P^{m,i}}{P^d}\right) i^m \left(\widehat{P}_t^{m,i} - \widehat{P}_t^d\right) - \left(\frac{SP^*}{P^d}\right) (c^m + i^m) \left(\widehat{S}_t + \widehat{P}_t^* - \widehat{P}_t^d\right) \end{aligned} \tag{182}$$

If we allow for a non-zero markup($\eta^{m,c} < \infty, \eta^{m,i} < \infty$) we need to introduce a fixed cost, $\phi^{m,c}$ and $\phi^{m,i}$, to ensure that profits are zero in steady state, or re-transfer the profits to the households. Imposing that the real profits for the importing consumption firm are zero in steady state implies the following;

$$\overline{\Pi}^{m,c} = \frac{P^{m,c}}{P^d} c^m - \frac{SP^*}{P^d} c^m - \phi^{m,c} = 0. \tag{183}$$

Inserting the steady state value of $P^{m,c} = \frac{\eta^{m,c}}{\eta^{m,c}-1} SP^*$, and using that $\frac{SP^*}{P^d} = 1$, yields that the value of $\phi^{m,c}$ must equal

$$\phi^{m,c} = \left(\frac{\eta^{m,c}}{\eta^{m,c} - 1} - 1\right) c^m, \tag{184}$$

$$\begin{aligned} \overline{\Pi}^{m,c} &= \frac{P^{m,c}}{P^d} c^m - \frac{SP^*}{P^d} c^m - \left(\frac{\eta^{m,c}}{\eta^{m,c} - 1} - 1\right) c^m && (185) \\ &= \frac{\left(\frac{\eta^{m,c}}{\eta^{m,c}-1} SP^*\right)}{P^d} c^m - c^m - \left(\frac{\eta^{m,c}}{\eta^{m,c} - 1} - 1\right) c^m && (186) \\ &= \left(\frac{\eta^{m,c}}{\eta^{m,c} - 1} - 1\right) c^m - \left(\frac{\eta^{m,c}}{\eta^{m,c} - 1} - 1\right) c^m && (187) \\ &= 0 && (188) \end{aligned}$$

and equivalently

$$\phi^{m,i} = \left(\frac{\eta^{m,i}}{\eta^{m,i}-1} - 1\right) i^{m}. \tag{189}$$

In this case the linearized profits must be

$$\begin{aligned}
\overline{\overline{\Pi}}_t^m &= \gamma_t^{mc,d} c_t^m + \gamma_t^{mi,d} i_t^m - \gamma_t^x \left(c_t^m + i_t^m\right) - \gamma_t^x \left(\phi^{m,c} + \phi^{m,i}\right) \\
&= \left(\gamma_t^{mc,d} - \gamma_t^x\right) c_t^m + \left(\gamma_t^{mi,d} - \gamma_t^x\right) i_t^m - \gamma_t^x \left(\phi^{m,c} + \phi^{m,i}\right),
\end{aligned}$$

The Taylor expansion, in this case, follows

$$\begin{aligned}
\widehat{\overline{\Pi}}_t^m &= c^m \left(\gamma^{mc,d} - \gamma^x\right) \widehat{c}_t^m + i^m \left(\gamma^{mi,d} - \gamma^x\right) \widehat{i}_t^m + \gamma^{mc,d} c^m \widehat{\gamma}_t^{mc,d} \\
&\quad + \gamma^{mi,d} i^m \widehat{\gamma}_t^{mi,d} - \gamma^x (c^m + i^m + \phi^{m,c} + \phi^{m,i}) \widehat{\gamma}_t^x \\
&= c^m \left(\gamma^{mc,d} - \gamma^x\right) \widehat{c}_t^m + i^m \left(\gamma^{mi,d} - \gamma^x\right) \widehat{i}_t^m + \gamma^{mc,d} c^m \widehat{\gamma}_t^{mc,d} \\
&\quad + \gamma^{mi,d} i^m \widehat{\gamma}_t^{mi,d} - \gamma^x \left(\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right) c^m + \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right) i^m\right) \widehat{\gamma}_t^x
\end{aligned} \tag{190}$$

or

$$\begin{aligned}
\widehat{\overline{\Pi}}_t^m &= c^m \left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right) \widehat{c}_t^m + i^m \left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right) \widehat{i}_t^m + \gamma^{mc,d} c^m \widehat{\gamma}_t^{mc,d} \\
&\quad + \gamma^{mi,d} i^m \widehat{\gamma}_t^{mi,d} + \frac{1}{\gamma^f} (c^m + i^m + \phi^{m,c} + \phi^{m,i}) \widehat{\gamma}_t^f \\
&= c^m \left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right) \widehat{c}_t^m + i^m \left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right) \widehat{i}_t^m + \gamma^{mc,d} c^m \widehat{\gamma}_t^{mc,d} \\
&\quad + \gamma^{mi,d} i^m \widehat{\gamma}_t^{mi,d} + \frac{1}{\gamma^f} \left(\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right) c^m + \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right) i^m\right) \widehat{\gamma}_t^f
\end{aligned} \tag{191}$$

4.3 Total

So, from using the log-linearization of equation (175) and noting that $\gamma^x = \frac{1}{\gamma^f}$ and $\widehat{\gamma}_t^x = -\widehat{\gamma}_t^f$, total profits follow

$$
\begin{aligned}
\widehat{\bar{\Pi}}_t &= \widehat{\bar{\Pi}}_t^d + \widehat{\bar{\Pi}}_t^m \\
&= y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\widehat{y}_t - \frac{1}{\lambda_f}(y+\phi)\widehat{mc}_t \\
&\quad + c^m\left(\gamma^{mc,d} - \gamma^x\right)\widehat{c}_t^m + i^m\left(\gamma^{mi,d} - \gamma^x\right)\widehat{i}_t^m + \gamma^{mc,d}c^m\widehat{\gamma}_t^{mc,d} \\
&\quad + \gamma^{mi,d}i^m\widehat{\gamma}_t^{mi,d} - \gamma^x\left(\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)c^m + \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)i^m\right)\widehat{\gamma}_t^x
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\bar{\Pi}}_t &= \widehat{\bar{\Pi}}_t^d + \widehat{\bar{\Pi}}_t^m \\
&= y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\left(\lambda_f(1-\alpha)\hat{H}_t + \lambda_f\alpha\left(\hat{k}_t - \hat{\mu}_{z,t}\right) + \lambda_f\hat{\epsilon}_t\right) - \frac{1}{\lambda_f}(y+\phi)\left(\alpha\left(\hat{\mu}_{z,t} + \hat{H}_t - \hat{k}_t\right) + \widehat{\bar{w}}_t + \hat{R}_t^f - \right. \\
&\quad + c^m\left(\gamma^{mc,d} - \gamma^x\right)\left(\eta_c\widehat{\gamma}_t^{c,mc} + \widehat{c}_t\right) + i^m\left(\gamma^{mi,d} - \gamma^x\right)\left(\eta_i\widehat{\gamma}_t^{i,mi} + \widehat{i}_t\right) + \gamma^{mc,d}c^m\widehat{\gamma}_t^{mc,d} \\
&\quad + \gamma^{mi,d}i^m\widehat{\gamma}_t^{mi,d} + \frac{1}{\gamma^f}\left(\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)c^m + \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)i^m\right)\widehat{\gamma}_t^f
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\bar{\Pi}}_t &= \widehat{\bar{\Pi}}_t^d + \widehat{\bar{\Pi}}_t^m \\
&= y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f(1-\alpha)\hat{H}_t + y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f\alpha\hat{k}_t - y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f\alpha\hat{\mu}_{z,t} + y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f\hat{\epsilon}_t \\
&\quad - \frac{1}{\lambda_f}(y+\phi)\alpha\hat{\mu}_{z,t} - \frac{1}{\lambda_f}(y+\phi)\alpha\hat{H}_t + \frac{1}{\lambda_f}(y+\phi)\alpha\hat{k}_t - \frac{1}{\lambda_f}(y+\phi)\widehat{\bar{w}}_t - \frac{1}{\lambda_f}(y+\phi)\hat{R}_t^f + \frac{1}{\lambda_f}(y+\phi)\hat{\epsilon}_t \\
&\quad + c^m\left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right)\eta_c\left(-(1-\omega_c)\left(\frac{1}{\gamma^{c,mc}\gamma^{mc,d}}\right)^{1-\eta_c}\right)\widehat{\gamma}_t^{mc,d} + c^m\left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right)\widehat{c}_t \\
&\quad + i^m\left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right)\eta_i\left(-(1-\omega_i)\left(\frac{1}{\gamma^{i,mi}\gamma^{mi,d}}\right)^{1-\eta_i}\right)\widehat{\gamma}_t^{mi,d} + i^m\left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right)\widehat{i}_t \\
&\quad + \gamma^{mc,d}c^m\widehat{\gamma}_t^{mc,d} + \gamma^{mi,d}i^m\widehat{\gamma}_t^{mi,d} \\
&\quad + \frac{1}{\gamma^f}\left(\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)c^m + \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)i^m\right)\widehat{\gamma}_t^f
\end{aligned}
$$

$$\begin{aligned}
\widehat{\overline{\Pi}}_t &= \widehat{\overline{\Pi}}_t^d + \widehat{\overline{\Pi}}_t^m \\
&= y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f\left(1-\alpha\right)\hat{H}_t + y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f\alpha\hat{k}_t - y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f\alpha\hat{\mu}_{z,t} + y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\lambda_f\hat{\epsilon}_t \\
&\quad - \frac{1}{\lambda_f}(y+\phi)\alpha\hat{\mu}_{z,t} - \frac{1}{\lambda_f}(y+\phi)\alpha\hat{H}_t + \frac{1}{\lambda_f}(y+\phi)\alpha\hat{k}_t - \frac{1}{\lambda_f}(y+\phi)\widehat{\overline{w}}_t - \frac{1}{\lambda_f}(y+\phi)\hat{R}_t^f + \frac{1}{\lambda_f}(y+\phi)\hat{\epsilon}_t \\
&\quad + \left[c^m\left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right)\eta_c\left(-(1-\omega_c)\left(\frac{1}{\gamma^{c,mc}\gamma^{mc,d}}\right)^{1-\eta_c}\right) + \gamma^{mc,d}c^m\right]\widehat{\gamma}_t^{mc,d} + c^m\left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right)\widehat{c}_t \\
&\quad + \left[i^m\left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right)\eta_i\left(-(1-\omega_i)\left(\frac{1}{\gamma^{i,mi}\gamma^{mi,d}}\right)^{1-\eta_i}\right) + \gamma^{mi,d}i^m\right]\widehat{\gamma}_t^{mi,d} + i^m\left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right)\widehat{i}_t \\
&\quad + \frac{1}{\gamma^f}\left(\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)c^m + \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)i^m\right)\widehat{\gamma}_t^f
\end{aligned}$$

$$\begin{aligned}
\widehat{\overline{\Pi}}_t &= \widehat{\overline{\Pi}}_t^d + \widehat{\overline{\Pi}}_t^m \\
&= y\left(\frac{\lambda_f - 1}{\lambda_f}\right)\widehat{y}_t - \frac{1}{\lambda_f}(y+\phi)\left(\alpha\left(\hat{\mu}_{z,t} + \hat{H}_t - \hat{k}_t\right) + \widehat{\overline{w}}_t + \hat{R}_t^f - \hat{\epsilon}_t\right) \\
&\quad + \left(c^m\left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right)\eta_c\left(-(1-\omega_c)\left(\frac{1}{\gamma^{c,mc}\gamma^{mc,d}}\right)^{1-\eta_c}\right) + \gamma^{mc,d}c^m\right)\widehat{\gamma}_t^{mc,d} + c^m\left(\gamma^{mc,d} - \frac{1}{\gamma^f}\right)\widehat{c}_t \\
&\quad + \left(i^m\left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right)\eta_i\left(-(1-\omega_i)\left(\frac{1}{\gamma^{i,mi}\gamma^{mi,d}}\right)^{1-\eta_i}\right) + \gamma^{mi,d}i^m\right)\widehat{\gamma}_t^{mi,d} + i^m\left(\gamma^{mi,d} - \frac{1}{\gamma^f}\right)\widehat{i}_t \\
&\quad + \frac{1}{\gamma^f}\left(\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)c^m + \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)i^m\right)\widehat{\gamma}_t^f. \qquad (192)
\end{aligned}$$

4.4 Linearization of the consumption aggregate

The CES function for aggregate consumption follows

$$C_t = \left[(1-\omega_c)^{\frac{1}{\eta_c}}\left(C_t^d\right)^{\frac{(\eta_c-1)}{\eta_c}} + \omega_c^{\frac{1}{\eta_c}}\left(C_t^m\right)^{\frac{(\eta_c-1)}{\eta_c}}\right]^{\frac{\eta_c}{(\eta_c-1)}}.$$

Stationarizing

$$\frac{C_t}{z_t} = \left[(1-\omega_c)^{\frac{1}{\eta_c}}\left(\frac{C_t^d}{z_t}\right)^{\frac{(\eta_c-1)}{\eta_c}} + \omega_c^{\frac{1}{\eta_c}}\left(\frac{C_t^m}{z_t}\right)^{\frac{(\eta_c-1)}{\eta_c}}\right]^{\frac{\eta_c}{(\eta_c-1)}},$$

$$F = c_t - \left[(1-\omega_c)^{\frac{1}{\eta_c}} \left(c_t^d\right)^{\frac{(\eta_c-1)}{\eta_c}} + \omega_c^{\frac{1}{\eta_c}} \left(c_t^m\right)^{\frac{(\eta_c-1)}{\eta_c}} \right]^{\frac{\eta_c}{(\eta_c-1)}} = 0.$$

Differentiate this expression with respect to c_t, c_t^d, c_t^m, evaluating the derivatives in steady state.

$$\frac{\partial F}{\partial c_t} = 1.$$

$$\begin{aligned} \frac{\partial F}{\partial c_t^d} &= -\frac{\eta_c}{(\eta_c-1)} [.]^{\frac{\eta_c}{(\eta_c-1)}-1} \frac{(\eta_c-1)}{\eta_c} (1-\omega_c)^{\frac{1}{\eta_c}} \left(c_t^d\right)^{\frac{(\eta_c-1)}{\eta_c}-1} \\ &= -c^{\frac{1}{\eta_c}} (1-\omega_c)^{\frac{1}{\eta_c}} \left(c^d\right)^{\frac{(\eta_c-1)}{\eta_c}-1}. \end{aligned}$$

Note that $[.]^{\frac{\eta_c}{(\eta_c-1)}-1} = \left(c^{\frac{(\eta_c-1)}{\eta_c}}\right)^{\frac{\eta_c}{(\eta_c-1)}-1} = \left(c^{\frac{(\eta_c-1)}{\eta_c}}\right)^{\frac{1}{(\eta_c-1)}} = c^{\frac{1}{\eta_c}}$.

$$\begin{aligned} \frac{\partial F}{\partial c_t^m} &= -\frac{\eta_c}{(\eta_c-1)} [.]^{\frac{\eta_c}{(\eta_c-1)}-1} \frac{(\eta_c-1)}{\eta_c} \omega_c^{\frac{1}{\eta_c}} \left(c_t^m\right)^{\frac{(\eta_c-1)}{\eta_c}-1} \\ &= -c^{\frac{1}{\eta_c}} \omega_c^{\frac{1}{\eta_c}} \left(c^m\right)^{\frac{(\eta_c-1)}{\eta_c}-1}. \end{aligned}$$

The Taylor expansion will follow:

$$0 = c\widehat{c}_t + c^d \left(-c^{\frac{1}{\eta_c}} (1-\omega_c)^{\frac{1}{\eta_c}} \left(c^d\right)^{\frac{(\eta_c-1)}{\eta_c}-1} \right) \widehat{c}_t^d + c^m \left(-c^{\frac{1}{\eta_c}} \omega_c^{\frac{1}{\eta_c}} \left(c^m\right)^{\frac{(\eta_c-1)}{\eta_c}-1} \right) \widehat{c}_t^m.$$

Solving for $\widehat{c}_t$

$$\widehat{c}_t = \left((1-\omega_c)^{\frac{1}{\eta_c}} \left(\frac{c^d}{c}\right)^{\frac{(\eta_c-1)}{\eta_c}} \right) \widehat{c}_t^d + \left(\omega_c^{\frac{1}{\eta_c}} \left(\frac{c^m}{c}\right)^{\frac{(\eta_c-1)}{\eta_c}} \right) \widehat{c}_t^m.$$

4.5 Linearization of the price aggregate

The aggregate price index follows

$$F = P_t^c - \left[(1-\omega_c) \left(P_t^d\right)^{1-\eta_c} + \omega_c \left(P_t^{m,c}\right)^{1-\eta_c} \right]^{\frac{1}{1-\eta_c}} = 0.$$

Differentiate with respect to P_t^c, P_t^d, and, $P_t^{m,c}$:

$$\frac{\partial F}{\partial P_t^c} = 1.$$

$$\begin{aligned}
\frac{\partial F}{\partial P_t^d} &= -\frac{1}{1-\eta_c}\left[.\right]^{\frac{1}{1-\eta_c}-1}(1-\omega_c)(1-\eta_c)\left(P^d\right)^{-\eta_c} \\
&= -\left((P^c)^{1-\eta_c}\right)^{\frac{\eta_c}{1-\eta_c}}(1-\omega_c)\left(P^d\right)^{-\eta_c}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F}{\partial P_t^m} &= -\frac{1}{1-\eta_c}\left[.\right]^{\frac{1}{1-\eta_c}-1}(\omega_c)(1-\eta_c)\left(P^{m,c}\right)^{-\eta_c} \\
&= -\left((P^c)^{1-\eta_c}\right)^{\frac{\eta_c}{1-\eta_c}}(\omega_c)\left(P^{m,c}\right)^{-\eta_c}.
\end{aligned}$$

The Taylor expansion follows

$$\begin{aligned}
0 &= P^c\widehat{P}_t^c + P^d\left(-\left((P^c)^{1-\eta_c}\right)^{\frac{\eta_c}{1-\eta_c}}(1-\omega_c)\left(P^d\right)^{-\eta_c}\right)\widehat{P}_t^d \\
&\quad + P_t^m\left(-\left((P^c)^{1-\eta_c}\right)^{\frac{\eta_c}{1-\eta_c}}(\omega_c)\left(P^{m,c}\right)^{-\eta_c}\right)\widehat{P}_t^m \\
&= P^c\widehat{P}_t^c - \left((P^c)^{\eta_c}(1-\omega_c)\left(P^d\right)^{1-\eta_c}\right)\widehat{P}_t^d - \left((P^c)^{\eta_c}(\omega_c)\left(P^{m,c}\right)^{1-\eta_c}\right)\widehat{P}_t^{m,c}.
\end{aligned}$$

Solving for $\widehat{P}_t^c$

$$\widehat{P}_t^c = \left((1-\omega_c)\left(\frac{P^d}{P^c}\right)^{1-\eta_c}\right)\widehat{P}_t^d + \left((\omega_c)\left(\frac{P^{m,c}}{P^c}\right)^{1-\eta_c}\right)\widehat{P}_t^{m,c}.$$

Taking first differences:

$$\begin{aligned}
\widehat{\pi}_t^c &= \left((1-\omega_c)\left(\frac{P^d}{P^c}\right)^{1-\eta_c}\right)\widehat{\pi}_t^d + \left((\omega_c)\left(\frac{P^{m,c}}{P^c}\right)^{1-\eta_c}\right)\widehat{\pi}_t^{m,c}. \\
&= \left((1-\omega_c)\left(\gamma^{d,c}\right)^{1-\eta_c}\right)\widehat{\pi}_t^d + \left((\omega_c)\left(\gamma^{mc,c}\right)^{(1-\eta_c)}\right)\widehat{\pi}_t^{m,c}
\end{aligned}$$

5 Relative prices

First, note that the steady state relative price between aggregate consumption and imported consumption goods, and aggregate consumption and domestic goods is, respectively

$$\gamma^{c,mc} = \frac{P^c}{P^{m,c}} = \left[(1-\omega_c)\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{-(1-\eta_c)} + \omega_c\right]^{1/(1-\eta_c)},$$

$$\gamma^{c,d} = \frac{P^c}{P^d} = \left[(1-\omega_c) + \omega_c\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}\right]^{1/(1-\eta_c)},$$

and the steady state relative price between imported and domestic consumption is

$$\begin{aligned}
\gamma^{mc,d} &= \frac{P^{m,c}}{P^d} = \left[\frac{(1-\omega_c) + \omega_c\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}}{(1-\omega_c)\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{-(1-\eta_c)} + \omega_c}\right]^{1/(1-\eta_c)} \\
&= \left[\frac{(1-\omega_c) + \omega_c\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}}{\frac{1}{\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}}\left((1-\omega_c) + \omega_c\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{(1-\eta_c)}\right)}\right]^{1/(1-\eta_c)} \\
&= \left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right).
\end{aligned}$$

5.1 $\gamma_t^{c,d} = \frac{P_t^c}{P_t^d}$

Start by using the definition of the aggregate consumption price. Using this:

$$\begin{aligned}
\gamma_t^{c,d} &= \frac{P_t^c}{P_t^d} = \left[(1-\omega_c) + \omega_c\left(\frac{P_t^{m,c}}{P_t^d}\right)^{1-\eta_c}\right]^{\frac{1}{1-\eta_c}} \\
&= \left[(1-\omega_c) + \omega_c\left(\gamma_t^{mc,d}\right)^{1-\eta_c}\right]^{\frac{1}{1-\eta_c}}.
\end{aligned}$$

Differentiate this:

$$\frac{\partial}{\partial\gamma_t^{c,d}} = 1.$$

$$\frac{\partial}{\partial\gamma_t^{mc,d}} = \frac{1}{1-\eta_c}[.]^{\frac{1}{1-\eta_c}-1}\omega_c(1-\eta_c)\left(\gamma^{mc,d}\right)^{-\eta_c}.$$

The Taylor expansion follows

$$\begin{aligned}
\gamma^{c,d}\widehat{\gamma}_t^{c,d} &= \gamma^{mc,d}\frac{1}{1-\eta_c}\left[.\right]^{\frac{1}{1-\eta_c}-1}\omega_c\left(1-\eta_c\right)\left(\gamma^{mc,d}\right)^{-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= \left[.\right]^{\frac{1}{1-\eta_c}-1}\omega_c\left(\gamma^{mc,d}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= \left(\left(\gamma^{c,d}\right)^{1-\eta_c}\right)^{\frac{\eta_c}{1-\eta_c}}\omega_c\left(\gamma^{mc,d}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d}.
\end{aligned}$$

Solving for $\widehat{\gamma}_t^{cd}$:

$$\begin{aligned}
\widehat{\gamma}_t^{c,d} &= \omega_c\left(\frac{\gamma^{mc,d}}{\gamma^{c,d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= \omega_c\left(\frac{\frac{P_t^{m,c}}{P_t^d}}{\frac{P_t^c}{P_t^d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= \omega_c\left(\frac{P_t^{m,c}}{P_t^c}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= \omega_c\left(\frac{1}{\frac{P_t^c}{P_t^{m,c}}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= \omega_c\left(\frac{1}{\gamma^{c,mc}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= \omega_c\left(\gamma^{c,mc}\right)^{-(1-\eta_c)}\widehat{\gamma}_t^{mc,d}.
\end{aligned}$$

$$\widehat{\gamma}_t^{d,c} = -\omega_c\left(\gamma^{c,mc}\right)^{-(1-\eta_c)}\widehat{\gamma}_t^{mc,d}$$

5.2 $\gamma_t^{c,mc} = \frac{P_t^c}{P_t^{m,c}}$

The consumption/import relative price is:

$$
\begin{aligned}
\gamma_t^{c,mc} &= \frac{P_t^c}{P_t^{m,c}} = \left[(1-\omega_c)\left(\frac{P_t^d}{P_t^{m,c}}\right)^{1-\eta_c} + \omega_c\right]^{\frac{1}{1-\eta_c}} \\
&= \left[(1-\omega_c)\left(\frac{1}{\gamma_t^{mc,d}}\right)^{1-\eta_c} + \omega_c\right]^{\frac{1}{1-\eta_c}}.
\end{aligned}
$$

Differentiate this with respect to γ_t^{cm} and γ_t^{md}:

$$
\frac{\partial}{\partial \gamma_t^{c,mc}} = 1.
$$

$$
\begin{aligned}
\frac{\partial}{\partial \gamma_t^{mc,d}} &= \frac{1}{1-\eta_c}\,[.]^{\frac{1}{1-\eta_c}-1}\,(1-\omega_c)(1-\eta_c)\left(\frac{1}{\gamma^{mc,d}}\right)^{-\eta_c}\frac{-1}{\left(\gamma^{mc,d}\right)^2} \\
&= \left(\left(\gamma^{c,mc}\right)^{1-\eta_c}\right)^{\frac{\eta_c}{1-\eta_c}}(1-\omega_c)\left(\frac{1}{\gamma^{mc,d}}\right)^{-\eta_c}\frac{-1}{\left(\gamma^{mc,d}\right)^2}.
\end{aligned}
$$

The Taylor expansion follows

$$
\begin{aligned}
\gamma^{c,mc}\widehat{\gamma}_t^{c,mc} &= \gamma^{mc,d}\left(\left(\gamma^{c,mc}\right)^{1-\eta_c}\right)^{\frac{\eta_c}{1-\eta_c}}(1-\omega_c)\left(\frac{1}{\gamma^{mc,d}}\right)^{-\eta_c}\frac{-1}{\left(\gamma^{mc,d}\right)^2}\widehat{\gamma}_t^{mc,d} \\
&= -\left(\gamma^{c,mc}\right)^{\eta_c}(1-\omega_c)\left(\frac{1}{\gamma^{mc,d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d}.
\end{aligned}
$$

Solving for $\widehat{\gamma}_t^{c,mc}$;

$$
\begin{aligned}
\widehat{\gamma}_t^{c,mc} &= -\left(\gamma^{c,mc}\right)^{\eta_c - 1}(1-\omega_c)\left(\frac{1}{\gamma^{mc,d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= -\left(\gamma^{c,mc}\right)^{-(1-\eta_c)}(1-\omega_c)\left(\frac{1}{\gamma^{mc,d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= -(1-\omega_c)\left(\frac{1}{\gamma^{c,mc}\gamma^{mc,d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d}. \\
&= -(1-\omega_c)\left(\frac{1}{\frac{P^c}{P^{m,c}}\frac{P^{m,c}}{P^d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= -(1-\omega_c)\left(\frac{1}{\frac{P^c}{P^d}}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= -(1-\omega_c)\left(\frac{P^d}{P^c}\right)^{1-\eta_c}\widehat{\gamma}_t^{mc,d} \\
&= -(1-\omega_c)\left(\frac{P^c}{P^d}\right)^{-(1-\eta_c)}\widehat{\gamma}_t^{mc,d} \\
&= -(1-\omega_c)\left(\gamma^{c,d}\right)^{-(1-\eta_c)}\widehat{\gamma}_t^{mc,d}
\end{aligned}
$$

5.3 $\gamma_t^{i,d} = \frac{P_t^i}{P_t^d}$

The imported investment relative prices also follow from the price aggregates:

$$
\begin{aligned}
\gamma_t^{i,d} &= \frac{P_t^i}{P_t^d} = \left[(1-\omega_i) + \omega_i\left(\frac{P_t^{m,i}}{P_t^d}\right)^{1-\eta_i}\right]^{\frac{1}{1-\eta_i}} \\
&= \left[(1-\omega_i) + \omega_i\left(\gamma_t^{mi,d}\right)^{1-\eta_i}\right]^{\frac{1}{1-\eta_i}}.
\end{aligned}
$$

$$
\begin{aligned}
\gamma_t^{i,mi} &= \frac{P_t^i}{P_t^{m,i}} = \left[(1-\omega_i)\left(\frac{P_t^d}{P_t^{m,i}}\right)^{1-\eta_i} + \omega_i\right]^{\frac{1}{1-\eta_i}} \\
&= \left[(1-\omega_i)\left(\frac{1}{\gamma_t^{mi,d}}\right)^{1-\eta_i} + \omega_i\right]^{\frac{1}{1-\eta_i}}.
\end{aligned}
$$

Consequently, and equivalently to the above, the log-linearized expressions follow:

$$
\begin{aligned}
\widehat{\gamma}_t^{i,d} &= \omega_i \left(\frac{1}{\gamma^{i,mi}}\right)^{1-\eta_i} \widehat{\gamma}_t^{mi,d} \\
&= \omega_i \left(\gamma^{i,mi}\right)^{-(1-\eta_i)} \widehat{\gamma}_t^{mi,d},
\end{aligned}
$$

and

$$
\widehat{\gamma}_t^{i,mi} = -(1-\omega_i) \left(\frac{1}{\gamma^{i,mi}\gamma^{mi,d}}\right)^{1-\eta_i} \widehat{\gamma}_t^{mi,d}.
$$

5.4 $\widehat{\gamma}_t^{mc,d}$

Second, define the domestic consumption 'terms of trade' as;

$$
\widehat{\gamma}_t^{mc,d} \equiv \widehat{P}_t^{m,c} - \widehat{P}_t^{d}. \tag{193}
$$

Lag this definitional equation and subtract the lagged expression from the equation above:

$$
\widehat{\gamma}_t^{mc,d} - \widehat{\gamma}_{t-1}^{mc,d} = \widehat{P}_t^{m,c} - \widehat{P}_t^{d} - \left(\widehat{P}_{t-1}^{m,c} - \widehat{P}_{t-1}^{d}\right),
$$

and consequently,

$$
\widehat{\gamma}_t^{mc,d} = \widehat{\gamma}_{t-1}^{mc,d} + \widehat{\pi}_t^{m,c} - \widehat{\pi}_t^{d}. \tag{194}
$$

5.5 $\widehat{\gamma}_t^{mi,d}$

The domestic investment 'terms of trade' is defined as

$$
\widehat{\gamma}_t^{mi,d} \equiv \widehat{P}_t^{m,i} - \widehat{P}_t^{d}. \tag{195}
$$

Similarly, lagging and subtracting yields:

$$
\widehat{\gamma}_t^{mi,d} = \widehat{\gamma}_{t-1}^{mi,d} + \widehat{\pi}_t^{m,i} - \widehat{\pi}_t^{d}. \tag{196}
$$

5.6 γ_t^f

The foreign terms of trade is defined as

$$\gamma_t^f \equiv \frac{P_t^d}{S_t P_t^*}$$

$$\begin{aligned}
\gamma_t^f &\equiv \frac{P_t^d}{S_t P_t^*} \\
\frac{\partial \gamma_t^f}{\partial \gamma_t^f} &= 1 \\
\frac{\partial \left(\frac{P_t^d}{S_t P_t^*}\right)}{\partial P_t^d} &= \frac{1}{S_t P_t^*} \\
\frac{\partial \left(\frac{P_t^d}{S_t P_t^*}\right)}{\partial S_t} &= -\frac{P_t^d}{S_t^2 P_t^*} \\
\frac{\partial \left(\frac{P_t^d}{S_t P_t^*}\right)}{\partial P_t^*} &= -\frac{P_t^d}{S_t \left(P_t^*\right)^2}
\end{aligned}$$

$$\begin{aligned}
\gamma_t^f \hat{\gamma}_t^f &\equiv P_t^d \frac{1}{S_t P_t^*} \hat{P}_t^d - S_t \frac{P_t^d}{S_t^2 P_t^*} \hat{S}_t - P_t^* \frac{P_t^d}{S_t \left(P_t^*\right)^2} \hat{P}_t^* \\
\frac{P_t^d}{S_t P_t^*} \hat{\gamma}_t^f &\equiv P_t^d \frac{1}{S_t P_t^*} \hat{P}_t^d - \frac{P_t^d}{S_t P_t^*} \hat{S}_t - \frac{P_t^d}{S_t \left(P_t^*\right)} \hat{P}_t^* \\
\hat{\gamma}_t^f &\equiv \hat{P}_t^d - \hat{S}_t - \hat{P}_t^*
\end{aligned}$$

$$\widehat{\gamma}_t^f \equiv \widehat{P}_t^d - \widehat{S}_t - \widehat{P}_t^*. \tag{197}$$

Lagging and subtracting yields:

$$\begin{aligned}
\widehat{\gamma}_t^f - \widehat{\gamma}_{t-1}^f &\equiv \left(\hat{P}_t^d - \hat{S}_t - \hat{P}_t^*\right) - \left(\hat{P}_{t-1}^d - \hat{S}_{t-1} - \hat{P}_{t-1}^*\right) & (198) \\
&\equiv \left(\hat{P}_t^d - \hat{P}_{t-1}^d\right) - \left(\hat{S}_t - \hat{S}_{t-1}\right) - \left(\hat{P}_t^* - \hat{P}_{t-1}^*\right) & (199)
\end{aligned}$$

$$\widehat{\gamma}_t^f \equiv \widehat{\gamma}_{t-1}^f + \widehat{\pi}_t^d - \Delta \widehat{S}_t - \widehat{\pi}_t^*. \tag{200}$$

The deviations from the law of one price for the imported consumption good (or, equivalently, the real marginal costs) can, from equations (**??**) and (193), be written as:

$$\begin{aligned} \widehat{mc}_t^{m,c} &= \widehat{S}_t + \widehat{P}_t^* - \widehat{P}_t^{m,c} = -\left(\widehat{P}_t^d - \widehat{S}_t - \widehat{P}_t^*\right) + \widehat{P}_t^d - \widehat{P}_t^{m,c} \\ &= -\widehat{\gamma}_t^f - \widehat{\gamma}_t^{mc,d}. \end{aligned} \tag{201}$$

Or equivalently,

$$\widehat{mc}_t^{m,c} = \widehat{mc}_{t-1}^{m,c} + \Delta \widehat{S}_t + \widehat{\pi}_t^* - \widehat{\pi}_t^{m,c}. \tag{202}$$

Similarly, the deviations from the law of one price for investment goods follows:

$$\widehat{mc}_t^{m,c} = -\widehat{\gamma}_t^f - \widehat{\gamma}_t^{mi,d}, \tag{203}$$

which implies,

$$\widehat{mc}_t^{m,i} = \widehat{mc}_{t-1}^{m,i} + \Delta \widehat{S}_t + \widehat{\pi}_t^* - \widehat{\pi}_t^{m,i}. \tag{204}$$

The real exchange rate is defined as:

$$\widehat{x}_t = \widehat{S}_t + \widehat{P}_t^* - \widehat{P}_t^c. \tag{205}$$

again, lagging and subtracting yields:

$$\widehat{x}_t = \widehat{x}_{t-1} + \Delta \widehat{S}_t + \widehat{\pi}_t^* - \widehat{\pi}_t^c. \tag{206}$$

So, collecting all the relative prices in compact form;

$$\widehat{mc}_t^{m,c} = \widehat{mc}_{t-1}^{m,c} + \Delta\widehat{S}_t + \widehat{\pi}_t^* - \widehat{\pi}_t^{m,c}. \tag{207}$$

$$\widehat{mc}_t^{m,i} = \widehat{mc}_{t-1}^{m,i} + \Delta\widehat{S}_t + \widehat{\pi}_t^* - \widehat{\pi}_t^{m,i}. \tag{208}$$

$$\widehat{\gamma}_t^{mc,d} = \widehat{\gamma}_{t-1}^{mc,d} + \widehat{\pi}_t^{m,c} - \widehat{\pi}_t^{d}. \tag{209}$$

$$\widehat{\gamma}_t^{mi,d} = \widehat{\gamma}_{t-1}^{mi,d} + \widehat{\pi}_t^{m,i} - \widehat{\pi}_t^{d}. \tag{210}$$

$$\widehat{\gamma}_t^{f} \equiv \widehat{\gamma}_{t-1}^{f} + \widehat{\pi}_t^{d} - \Delta\widehat{S}_t - \widehat{\pi}_t^*. \tag{211}$$

$$\widehat{\gamma}_t^{c,d} = \omega_c \left(\gamma^{c,mc}\right)^{-(1-\eta_c)} \widehat{\gamma}_t^{mc,d} \tag{212}$$

$$\widehat{\gamma}_t^{c,mc} = -(1-\omega_c)\left(\frac{1}{\gamma^{c,mc}\gamma^{mc,d}}\right)^{1-\eta_c} \widehat{\gamma}_t^{mc,d}. \tag{213}$$

$$= -(1-\omega_c)\left(\gamma^{c,d}\right)^{-(1-\eta_c)} \widehat{\gamma}_t^{mc,d} \tag{214}$$

$$\widehat{\gamma}_t^{i,d} = \omega_i \left(\gamma^{i,mi}\right)^{-(1-\eta_i)} \widehat{\gamma}_t^{mi,d}, \tag{215}$$

$$\widehat{\gamma}_t^{i,mi} = -(1-\omega_i)\left(\frac{1}{\gamma^{i,mi}\gamma^{mi,d}}\right)^{1-\eta_i} \widehat{\gamma}_t^{mi,d} \tag{216}$$

$$\widehat{x}_t = \widehat{x}_{t-1} + \Delta\widehat{S}_t + \widehat{\pi}_t^* - \widehat{\pi}_t^{c}. \tag{217}$$

or in a more comprehensive notation;

$$\left(\widehat{S}_t + \widehat{P}^*_t - \widehat{P}^{m,c}_t\right) = \left(\widehat{S}_{t-1} + \widehat{P}^*_{t-1} - \widehat{P}^{m,c}_{t-1}\right) + \Delta\widehat{S}_t + \widehat{\pi}^*_t - \widehat{\pi}^{m,c}_t. \tag{218}$$

$$\left(\widehat{S}_t + \widehat{P}^*_t - \widehat{P}^{m,i}_t\right) = \left(\widehat{S}_{t-1} + \widehat{P}^*_{t-1} - \widehat{P}^{m,i}_{t-1}\right) + \Delta\widehat{S}_t + \widehat{\pi}^*_t - \widehat{\pi}^{m,i}_t. \tag{219}$$

$$\left(\widehat{P}^{m,c}_t - \widehat{P}^d_t\right) = \left(\widehat{P}^{m,c}_{t-1} - \widehat{P}^d_{t-1}\right) + \widehat{\pi}^{m,c}_t - \widehat{\pi}^d_t. \tag{220}$$

$$\left(\widehat{P}^{m,i}_t - \widehat{P}^d_t\right) = \left(\widehat{P}^{m,i}_{t-1} - \widehat{P}^d_{t-1}\right) + \widehat{\pi}^{m,i}_t - \widehat{\pi}^d_t. \tag{221}$$

$$\left(\widehat{P}^d_t - \widehat{S}_t - \widehat{P}^*_t\right) \equiv \left(\widehat{P}^d_{t-1} - \widehat{S}_{t-1} - \widehat{P}^*_{t-1}\right) + \widehat{\pi}^d_t - \Delta\widehat{S}_t - \widehat{\pi}^*_t \tag{222}$$

$$\left(\widehat{P}^c_t - \widehat{P}^d_t\right) = \omega_c \left(\frac{P^c}{P^{m,c}}\right)^{-(1-\eta_c)} \left(\widehat{P}^{m,c}_t - \widehat{P}^d_t\right) \tag{223}$$

$$\left(\widehat{P}^c_t - \widehat{P}^{m,c}_t\right) = -(1-\omega_c) \left(\frac{P^{mc}}{P^c} \frac{P^d}{P^{m,c}}\right)^{1-\eta_c} \left(\widehat{P}^{m,c}_t - \widehat{P}^d_t\right) \tag{224}$$

$$\left(\widehat{P}^i_t - \widehat{P}^d_t\right) = \omega_i \left(\frac{P^i}{P^{m,i}}\right)^{-(1-\eta_i)} \left(\widehat{P}^{m,i}_t - \widehat{P}^d_t\right). \tag{225}$$

$$\left(\widehat{P}^i_t - \widehat{P}^{m,i}_t\right) = -(1-\omega_i) \left(\frac{P^{mi}}{P^i} \frac{P^d}{P^{m,i}}\right)^{1-\eta_i} \left(\widehat{P}^{m,i}_t - \widehat{P}^d_t\right). \tag{226}$$

$$\left(\widehat{S}_t + \widehat{P}^*_t - \widehat{P}^c_t\right) = \left(\widehat{S}_{t-1} + \widehat{P}^*_{t-1} - \widehat{P}^c_{t-1}\right) + \Delta\widehat{S}_t + \widehat{\pi}^*_t - \widehat{\pi}^c_t \tag{227}$$

Evaluating these relative prices in steady state;

$$\begin{aligned}
P^{m,c} &= \frac{\eta^{m,c}}{\eta^{m,c}-1} SP^*. && (228)\\
P^{m,i} &= \frac{\eta^{m,i}}{\eta^{m,i}-1} SP^*. && (229)\\
\frac{P^{m,c}}{P^d} &= \left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right). && (230)\\
\frac{P^{m,i}}{P^d} &= \left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right). && (231)\\
\frac{P^d}{SP^*} &= 1, \text{ by definition.} && (232)\\
\frac{P^c}{P^d} &= \left[(1-\omega_c)+\omega_c\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}\right]^{1/(1-\eta_c)} && (233)\\
\frac{P^c}{P^{m,c}} &= \left[(1-\omega_c)\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{-(1-\eta_c)}+\omega_c\right]^{1/(1-\eta_c)}. && (234)\\
\frac{P^i}{P^d} &= \left[(1-\omega_i)+\omega_i\left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)^{1-\eta_i}\right]^{1/(1-\eta_i)}. && (235)\\
\frac{P^i}{P^{m,i}} &= \left[(1-\omega_i)\left(\frac{\eta^{m,i}}{\eta^{m,i}-1}\right)^{-(1-\eta_i)}+\omega_i\right]^{1/(1-\eta_i)}. && (236)\\
\frac{SP^*}{P^c} &= \frac{SP^*P^d}{P^cP^d} = \frac{P^d}{P^c} = \left[(1-\omega_c)+\omega_c\left(\frac{\eta^{m,c}}{\eta^{m,c}-1}\right)^{1-\eta_c}\right]^{-1/(1-\eta_c)} && (237)
\end{aligned}$$

5.7 Relative consumption

The relative consumption of imported consumption goods follows

$$\begin{aligned}
C_t^m &= \omega_c\left(\frac{P_t^{m,c}}{P_t^c}\right)^{-\eta_c} C_t\\
&= \omega_c\left(\gamma_t^{c,mc}\right)^{\eta_c} C_t.
\end{aligned}$$

Stationarize this:

$$\frac{C_t^m}{z_t} = \omega_c\left(\gamma_t^{c,mc}\right)^{\eta_c}\frac{C_t}{z_t},$$

or equivalently,

$$c_t^m = \omega_c \left(\gamma_t^{c,mc}\right)^{\eta_c} c_t.$$

Differentiate this with respect to c_t^m, γ_t^{cm}, and c_t, evaluating in steady state:

$$\frac{\partial .}{\partial c_t^m} = 1.$$

$$\frac{\partial .}{\partial \gamma_t^{c,mc}} = \omega_c \eta_c \left(\gamma^{c,mc}\right)^{\eta_c - 1} c.$$

$$\frac{\partial .}{\partial c_t} = \omega_c \left(\gamma^{c,mc}\right)^{\eta_c}.$$

In steady state we know that $c^m = \omega_c \left(\gamma_t^{c,mc}\right)^{\eta_c} c$.

The Taylor approximation is:

$$c^m \widehat{c}_t^m = \gamma^{c,mc} \omega_c \eta_c \left(\gamma^{c,mc}\right)^{\eta_c - 1} c \widehat{\gamma}_t^{c,mc} + c \omega_c \left(\gamma^{c,mc}\right)^{\eta_c} \widehat{c}_t.$$

$$\begin{aligned}
\widehat{c}_t^m &= \omega_c \eta_c \left(\gamma^{c,mc}\right)^{\eta_c} \frac{c}{c^m} \widehat{\gamma}_t^{c,mc} + \frac{c}{c^m} \omega_c \left(\gamma^{c,mc}\right)^{\eta_c} \widehat{c}_t \\
&= \omega_c \left(\gamma^{c,mc}\right)^{\eta_c} \frac{c}{\omega_c \left(\gamma_t^{c,mc}\right)^{\eta_c} c} \left(\eta_c \widehat{\gamma}_t^{c,mc} + \widehat{c}_t\right) \\
&= \eta_c \widehat{\gamma}_t^{c,mc} + \widehat{c}_t \\
&= \eta_c \left(\widehat{P}_t^c - \widehat{P}_t^{m,c}\right) + \widehat{c}_t. \qquad (238)
\end{aligned}$$

Similarly

$$\begin{aligned}
\widehat{c}_t^d &= \eta_c \widehat{\gamma}_t^{c,d} + \widehat{c}_t \\
&= \eta_c \left(\widehat{P}_t^c - \widehat{P}_t^d\right) + \widehat{c}_t. \qquad (239)
\end{aligned}$$

$$\begin{aligned}
\widehat{i}_t^d &= \eta_i \widehat{\gamma}_t^{i,d} + \widehat{i}_t \\
&= \eta_i \left(\widehat{P}_t^i - \widehat{P}_t^d\right) + \widehat{i}_t. \qquad (240)
\end{aligned}$$

$$
\begin{aligned}
\widehat{i}_t^{m} &= \eta_i \widehat{\gamma}_t^{i,mi} + \widehat{i}_t \\
&= \eta_i \left(\widehat{P}_t^{i} - \widehat{P}_t^{m,i} \right) + \widehat{i}_t. \qquad (241)
\end{aligned}
$$

6 The foreign demand

The foreign demand for the domestically produced goods follow a continuos CES function with the same elasticity of substitution as the domestic consumers face between domestic and imported goods. That is

$$
C_t^* = \left[\int_0^1 \left(C_{it}^* \right)^{\frac{\eta_c - 1}{\eta_c}} di \right]^{\frac{\eta_c - 1}{\eta_c}}.
$$

The demand for the domestic export good is consequently:

$$
\begin{aligned}
C_t^{df} &= \left(\frac{P_t^{df}}{P_t^*} \right)^{-\eta_c} C_t^* \\
&= \left(\frac{P_t^{d}}{S_t P_t^*} \right)^{-\eta_c} C_t^* \\
&= \left(\gamma_t^{f} \right)^{\eta_c} C_t^*.
\end{aligned}
$$

Log-linearizing this expression implies

$$
\widehat{C}_t^{df} = \eta_i \widehat{\gamma}_t^{f} + \widehat{C}_t^*. \qquad (242)
$$

$$
\widehat{c}_t^{df} = \eta_f \widehat{\gamma}_t^{f} + \widehat{c}_t^*. \qquad (243)
$$

$$
\widehat{i}_t^{df} = \eta_f \widehat{\gamma}_t^{f} + \widehat{i}_t^*. \qquad (244)
$$

7 Log-linearization of foc:s for $c_{\mathbf{t}}, m_{\mathbf{t+1}}, \bar{k}_{\mathbf{t+1}}, and$ i_t

7.1 Log-linearizing the consumption equation

The first-order condition for C_t is (make use of the fact that the households and average (aggregate) choices coincide in equilibrium)

$$\zeta_t^c u'\left(C_t - bC_{t-1}\right) - \beta b \mathrm{E}_t \zeta_{t+1}^c u'\left(C_{t+1} - bC_t\right) - \psi_t \frac{P_t^c}{P_t}\left(1 + \tau_t^c\right) = 0 \tag{245}$$

where we have introduced the notation

$$\psi_t \equiv \upsilon_t P_t. \tag{246}$$

Write out (247), taking into account $E_t E_t^{\mu} x = E_t x$:

$$E_t\left[-u'(C_t - bC_{t-1}) + b\beta u'(C_{t+1} - bC_t) + \psi_t\right] = 0. \tag{247}$$

Writing the object in square brackets, taking into account our log-utility assumption:

$$\begin{aligned} u'(C_t - bC_{t-1}) &= \\ \frac{\partial \ln\left(C_t - bC_{t-1}\right)}{\partial C_t} &= \frac{1}{C_t - bC_{t-1}} \\ u'\left(C_{t+1} - bC_t\right) &= \\ \frac{\partial \ln\left(C_{t+1} - bC_t\right)}{\partial C_t} &= \frac{b}{-C_{t+1} + bC_t} = -\frac{b}{\left(C_{t+1} - bC_t\right)} \end{aligned}$$

$$
\begin{aligned}
0 &= \zeta_t^c u'\left(C_t - bC_{t-1}\right) - \beta b E_t \zeta_{t+1}^c u'\left(C_{t+1} - bC_t\right) - \psi_t \frac{P_t^c}{P_t}\left(1+\tau_t^c\right) \\
&= \frac{\zeta_t^c}{C_t - bC_{t-1}} - \beta b E_t \frac{\zeta_{t+1}^c}{\left(C_{t+1} - bC_t\right)} - \psi_t \frac{P_t^c}{P_t}\left(1+\tau_t^c\right)
\end{aligned}
$$

Compare with $ACEL$:

$$
-\frac{1}{C_t - bC_{t-1}} + \frac{b\beta}{C_{t+1} - bC_t} + \psi_t
$$

Multiply by z_t :

$$
\begin{aligned}
& z_t \frac{\zeta_t^c}{C_t - bC_{t-1}} - z_t \beta b E_t \frac{\zeta_{t+1}^c}{\left(C_{t+1} - bC_t\right)} - z_t \psi_t \frac{P_t^c}{P_t}\left(1+\tau_t^c\right) \\
= \quad & \frac{\zeta_t^c}{\frac{C_t}{z_t} - b\frac{C_{t-1}}{z_t}} - \beta b E_t \frac{\zeta_{t+1}^c}{\frac{C_{t+1}}{z_t} - b\frac{C_t}{z_t}} - \psi_{z,t} \frac{P_t^c}{P_t}\left(1+\tau_t^c\right)
\end{aligned}
$$

where

$$
\psi_{z,t} = z_t \psi_t.
$$

Let

$$
c_t = \frac{C_t}{z_t}.
$$

Then,

$$
\begin{aligned}
& -\frac{1}{\frac{C_t}{z_t} - b\frac{C_{t-1}}{z_t}} + \frac{b\beta}{\frac{C_{t+1}}{z_t} - b\frac{C_t}{z_t}} + \psi_{z,t} \\
= \quad & -\frac{1}{c_t - b\frac{C_{t-1}}{z_{t-1}}\frac{z_{t-1}}{z_t}} + \frac{b\beta}{\frac{C_{t+1}}{z_{t+1}}\frac{z_{t+1}}{z_t} - b\frac{C_t}{z_t}} + \psi_{z,t} \\
= \quad & -\frac{1}{c_t - b\frac{C_{t-1}}{z_{t-1}}\frac{z_{t-1}}{z_t}} + \frac{b\beta}{\frac{C_{t+1}}{z_{t+1}}\frac{z_{t+1}}{z_t} - b\frac{C_t}{z_t}} + \psi_{z,t} \\
= \quad & -\frac{1}{c_t - bc_{t-1}\frac{1}{\mu_{z,t}}} + \frac{b\beta}{c_{t+1}\mu_{z,t+1} - bc_t} + \psi_{z,t}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{\zeta_t^c}{\frac{C_t}{z_t} - b\frac{C_{t-1}}{z_t}} + \beta b E_t \frac{\zeta_{t+1}^c}{\frac{C_{t+1}}{z_t} - b\frac{C_t}{z_t}} + \psi_{z,t}\frac{P_t^c}{P_t}\left(1+\tau_t^c\right) \\
= \quad & -\frac{\zeta_t^c}{\frac{C_t}{z_t} - b\frac{C_{t-1}}{z_{t-1}}\frac{z_{t-1}}{z_t}} + \beta b E_t \frac{\zeta_{t+1}^c}{\frac{C_{t+1}}{z_{t+1}}\frac{z_{t+1}}{z_t} - b\frac{C_t}{z_t}} + \psi_{z,t}\frac{P_t^c}{P_t}\left(1+\tau_t^c\right) \\
\Rightarrow \quad & \left\{\mu_{z,t+1} = \frac{z_{t+1}}{z_t} \Rightarrow \mu_{z,t} = \frac{z_t}{z_{t-1}} \Rightarrow \frac{1}{\mu_{z,t}} = \frac{z_{t-1}}{z_t}\right\} \Rightarrow \\
= \quad & \frac{\zeta_t^c}{c_t - bc_{t-1}\frac{1}{\mu_{z,t}}} + \beta b E_t \frac{\zeta_{t+1}^c}{c_{t+1}\mu_{z,t+1} - bc_t} + \psi_{z,t}\frac{P_t^c}{P_t}\left(1+\tau_t^c\right)
\end{aligned}
$$

So, (247) reduces to:

$$
\begin{aligned}
0 \quad = \quad & E_t\left[-u'(C_t - bC_{t-1}) + b\beta u'(C_{t+1} - bC_t) + \psi_t\right] && (248) \\
= \quad & E_t\left[\frac{\zeta_t^c}{c_t - bc_{t-1}\frac{1}{\mu_{z,t}}} + \beta b E_t \frac{\zeta_{t+1}^c}{c_{t+1}\mu_{z,t+1} - bc_t} + \psi_{z,t}\frac{P_t^c}{P_t}\left(1+\tau_t^c\right)\right] \\
= \quad & E_t\left[-\zeta_t^c\left(c_t - bc_{t-1}\frac{1}{\mu_{z,t}}\right)^{-1} + b\beta\zeta_{t+1}^c\left(c_{t+1}\mu_{z,t+1} - bc_t\right)^{-1} + \psi_{z,t}\frac{P_t^c}{P_t}\left(1+\tau_t^c\right)\right]
\end{aligned}
$$

Let's work out what $\psi_{z,t}$ is in steady state:

$$
E_t\left[\zeta_t^c\right] = 1 \qquad \frac{P^c}{P} = \gamma^c
$$

$$
0 \quad = \quad -\zeta\left(c - bc\frac{1}{\mu}\right)^{-1} + b\beta\zeta\left(c\mu - bc\right)^{-1} + \psi\gamma^c\left(1+\tau\right)
$$

$$
\begin{aligned}
\psi_z \quad = \quad & \frac{\zeta^c}{\left(1+\tau^c\right)\gamma^c}\left(c - bc\frac{1}{\mu_z}\right)^{-1} - \frac{b\beta\zeta^c}{\left(1+\tau^c\right)\gamma^c}\left(c\mu_z - bc\right)^{-1} \\
= \quad & \zeta^c\frac{\mu_z - b\beta}{c\left(1+\tau^c\right)\gamma^c\left(\mu_z - b\right)} \\
= \quad & \frac{1}{c\left(1+\tau^c\right)\gamma^c}\left(\frac{\mu_z - b\beta}{\mu_z - b}\right)
\end{aligned}
$$

Now let's totally differentiate the expression in square brackets in (248):

$$\begin{aligned} & F\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\ = \; & E_t\left[-\zeta_t^c\left(c_t-bc_{t-1}\frac{1}{\mu_{z,t}}\right)^{-1}+b\beta\zeta_{t+1}^c\left(c_{t+1}\mu_{z,t+1}-bc_t\right)^{-1}+\psi_{z,t}\gamma_t^c\left(1+\tau_t^c\right)\right]=0 \end{aligned}$$

$$\begin{aligned} & F_1\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\ = \; & -\left(c_t-bc_{t-1}\frac{1}{\mu_{z,t}}\right)^{-1} \\ -\left(c-bc\frac{1}{\mu_z}\right)^{-1} = \; & -\frac{\mu_z}{c\left(\mu_z-b\right)} \end{aligned}$$

$$\begin{aligned} & F_2\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\ = \; & b\beta\left(c_{t+1}\mu_{z,t+1}-bc_t\right)^{-1} \\ b\beta\left(c\mu_z-bc\right)^{-1} = \; & b\frac{\beta}{c\left(\mu_z-b\right)} \end{aligned}$$

$$\begin{aligned} & F_3\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\ = \; & \frac{\zeta_t^c}{\left(c_t-bc_{t-1}\frac{1}{\mu_{z,t}}\right)^2}+\frac{b^2\beta\zeta_{t+1}^c}{\left(-c_{t+1}\mu_{z,t+1}+bc_t\right)^2} \\ & \frac{1}{\left(c-bc\frac{1}{\mu_z}\right)^2}+\frac{b^2\beta}{\left(-c\mu_z+bc\right)^2} \\ = \; & \frac{\mu_z^2+b^2\beta}{c^2\left(\mu_z-b\right)^2}=\frac{1}{c^2}\left(\frac{\mu_z^2+b^2\beta}{\left(\mu_z-b\right)^2}\right) \end{aligned}$$

$$
\begin{aligned}
& F_4\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\
= \quad & -\zeta_t^c b \frac{\frac{1}{\mu_{z,t}}}{\left(-c_t + b c_{t-1}\frac{1}{\mu_{z,t}}\right)^2} \\
& -b \frac{\frac{1}{\mu_z}}{\left(-c + bc\frac{1}{\mu_z}\right)^2} \\
= \quad & -b \frac{\mu_z}{c^2\left(\mu_z - b\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
& F_5\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\
= \quad & -b\beta\zeta_{t+1}^c \frac{\mu_{z,t+1}}{\left(-c_{t+1}\mu_{z,t+1} + bc_t\right)^2} \\
& -b\beta \frac{\mu_z}{\left(-c\mu_z + bc\right)^2} \\
= \quad & -b\beta \frac{\mu_z}{c^2\left(\mu_z - b\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
& F_6\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\
= \quad & \gamma_t^c\left(1+\tau_t^c\right) \\
& \gamma^c\left(1+\tau^c\right)
\end{aligned}
$$

$$
\begin{aligned}
& F_7\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right) \\
= \quad & \psi_{z,t}\gamma_t^c \\
& \psi_z\gamma^c
\end{aligned}
$$

$$F_8\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right)$$

$$= \zeta_t^c \frac{bc_{t-1}}{\left(-c_t\mu_{z,t}+bc_{t-1}\right)^2}$$

$$\frac{bc}{\left(-c\mu_z+bc\right)^2} = \frac{b}{c\left(\mu_z-b\right)^2}$$

$$F_9\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right)$$

$$= -b\beta\zeta_{t+1}^c \frac{c_{t+1}}{\left(-c_{t+1}\mu_{z,t+1}+bc_t\right)^2}$$

$$-b\beta\frac{c}{\left(-c\mu_z+bc\right)^2} = -b\frac{\beta}{c\left(\mu_z-b\right)^2}$$

$$F_{10}\left(\zeta_t^c,\zeta_{t+1}^c,c_t,c_{t-1},c_{t+1},\psi_{z,t},\tau_t^c,\mu_{z,t},\mu_{z,t+1},\gamma_t^c\right)$$

$$= \psi_{z,t}\left(1+\tau_t^c\right)$$

$$\psi_z\left(1+\tau^c\right)$$

Collect terms and evaluate in steady state:

$$
\begin{aligned}
&-\frac{\mu_z}{c\left(\mu_z-b\right)}\hat{\zeta}_t^c \\
&+b\frac{\beta}{c\left(\mu_z-b\right)}\hat{\zeta}_{t+1}^c \\
&+\frac{1}{c^2}\left(\frac{\mu_z^2+b^2\beta}{\left(\mu_z-b\right)^2}\right)c\hat{c}_t \\
&-b\frac{\mu_z}{c^2\left(\mu_z-b\right)^2}c\hat{c}_{t-1} \\
&-b\beta\frac{\mu_z}{c^2\left(\mu_z-b\right)^2}c\hat{c}_{t+1} \\
&+\gamma^c\left(1+\tau^c\right)\psi_z\hat{\psi}_{z,t} \\
&+\psi_z\gamma^c\tau^c\hat{\tau}_t^c \\
&+\frac{b}{c\left(\mu_z-b\right)^2}\mu_z\hat{\mu}_{z,t} \\
&-b\frac{\beta}{c\left(\mu_z-b\right)^2}\mu_z\hat{\mu}_{z,t+1} \\
&+\psi_z\left(1+\tau^c\right)\gamma^c\hat{\gamma}_t^c
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{b\beta\mu_z}{c\left(\mu_z-b\right)^2}\hat{c}_{t+1}+\frac{1}{c}\left(\frac{\mu_z^2+b^2\beta}{\left(\mu_z-b\right)^2}\right)\hat{c}_t-b\frac{\mu_z}{c\left(\mu_z-b\right)^2}\hat{c}_{t-1}+\frac{b}{c\left(\mu_z-b\right)^2}\left(\mu_z\hat{\mu}_{z,t}-\beta\mu_z\hat{\mu}_{z,t+1}\right) \\
&+\gamma^c\left(1+\tau^c\right)\psi_z\hat{\psi}_{z,t}+\psi_z\gamma^c\tau^c\hat{\tau}_t^c+\psi_z\left(1+\tau^c\right)\gamma^c\hat{\gamma}_t^c-\frac{1}{c\left(\mu_z-b\right)}\left(\mu_z\hat{\zeta}_t^c-b\beta\hat{\zeta}_{t+1}^c\right)
\end{aligned}
$$

Multiply with $c\left(\mu_z-b\right)^2$

$$
\begin{aligned}
&-b\beta\mu_z\hat{c}_{t+1}+\left(\mu_z^2+b^2\beta\right)\hat{c}_t-b\mu_z\hat{c}_{t-1}+b\mu_z\left(\hat{\mu}_{z,t}-\beta\hat{\mu}_{z,t+1}\right) \\
&+c\left(\mu_z-b\right)^2\gamma^c\left(1+\tau^c\right)\psi_z\hat{\psi}_{z,t}+c\left(\mu_z-b\right)^2\psi_z\gamma^c\tau^c\hat{\tau}_t^c \\
&+c\left(\mu_z-b\right)^2\psi_z\left(1+\tau^c\right)\gamma^c\hat{\gamma}_t^c \\
&-\left(\mu_z-b\right)\left(\mu_z\hat{\zeta}_t^c-b\beta\hat{\zeta}_{t+1}^c\right) \qquad (**)
\end{aligned}
$$

Use

$$\psi_z = \frac{1}{c\left(1+\tau^c\right)\gamma^c}\left(\frac{\mu_z - b\beta}{\mu_z - b}\right)$$

to rewrite $(**)$

$$\begin{aligned}
&-b\beta\mu_z\hat{c}_{t+1} + \left(\mu_z^2 + b^2\beta\right)\hat{c}_t - b\mu_z\hat{c}_{t-1} + b\mu_z\left(\hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right) \\
&+c\left(\mu_z - b\right)^2\omega_c\gamma^c\left(1+\tau^c\right)\frac{1}{c\left(1+\tau^c\right)\omega_c\gamma^c}\left(\frac{\mu_z - b\beta}{\mu_z - b}\right)\hat{\psi}_{z,t} \\
&+c\left(\mu_z - b\right)^2\frac{1}{c\left(1+\tau^c\right)\omega_c\gamma^c}\left(\frac{\mu_z - b\beta}{\mu_z - b}\right)\omega_c\gamma^c\tau^c\hat{\tau}_t^c \\
&+c\left(\mu_z - b\right)^2\frac{1}{c\left(1+\tau^c\right)\omega_c\gamma^c}\left(\frac{\mu_z - b\beta}{\mu_z - b}\right)\omega_c\left(1+\tau^c\right)\gamma^c\hat{\gamma}_t^c \\
&-\left(\mu_z - b\right)\left(\mu_z\hat{\zeta}_t^c - b\beta\hat{\zeta}_{t+1}^c\right)
\end{aligned}$$

or

$$\begin{aligned}
&-b\beta\mu_z\hat{c}_{t+1} + \left(\mu_z^2 + b^2\beta\right)\hat{c}_t - b\mu_z\hat{c}_{t-1} + b\mu_z\left(\hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right) \\
&+\left(\mu_z - b\beta\right)\left(\mu_z - b\right)\hat{\psi}_{z,t} + \frac{\tau^c}{1+\tau^c}\left(\mu_z - b\beta\right)\left(\mu_z - b\right)\hat{\tau}_t^c \\
&+\left(\mu_z - b\beta\right)\left(\mu_z - b\right)\hat{\gamma}_t^c \\
&-\left(\mu_z - b\right)\left(\mu_z\hat{\zeta}_t^c - b\beta\hat{\zeta}_{t+1}^c\right)
\end{aligned}$$

or

$$
\begin{aligned}
\hat{c}_{t+1} &= \frac{\left(\mu_z^2 + b^2\beta\right)}{b\beta\mu_z}\hat{c}_t - \frac{b\mu_z}{b\beta\mu_z}\hat{c}_{t-1} + \frac{b\mu_z}{b\beta\mu_z}\left(\hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right) \\
&+ \frac{\left(\mu_z - b\beta\right)\left(\mu_z - b\right)}{b\beta\mu_z}\hat{\psi}_{z,t} + \frac{\tau^c}{1+\tau^c}\frac{\left(\mu_z - b\beta\right)\left(\mu_z - b\right)}{b\beta\mu_z}\hat{\tau}_t^c \\
&+ \frac{\left(\mu_z - b\beta\right)\left(\mu_z - b\right)}{b\beta\mu_z}\hat{\gamma}_t^c \\
&- \frac{\left(\mu_z - b\right)}{b\beta\mu_z}\left(\mu_z\hat{\zeta}_t^c - b\beta\hat{\zeta}_{t+1}^c\right)
\end{aligned}
$$

or

$$
\begin{aligned}
\hat{c}_{t+1} &= \frac{\left(\mu_z^2 + b^2\beta\right)}{b\beta\mu_z}\hat{c}_t - \frac{1}{\beta}\hat{c}_{t-1} + \frac{1}{\beta}\left(\hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right) \\
&+ \frac{\left(\mu_z - b\beta\right)\left(\mu_z - b\right)}{b\beta\mu_z}\hat{\psi}_{z,t} + \frac{\tau^c}{1+\tau^c}\frac{\left(\mu_z - b\beta\right)\left(\mu_z - b\right)}{b\beta\mu_z}\hat{\tau}_t^c \\
&+ \frac{\left(\mu_z - b\beta\right)\left(\mu_z - b\right)}{b\beta\mu_z}\hat{\gamma}_t^c \\
&- \frac{\left(\mu_z - b\right)}{b\beta\mu_z}\left(\mu_z\hat{\zeta}_t^c - b\beta\hat{\zeta}_{t+1}^c\right)
\end{aligned}
$$

Thus, in linearized form, (248) reduces to:

$$
E_t\left[\begin{array}{c}
-b\beta\mu_z\hat{c}_{t+1} + \left(\mu_z^2 + b^2\beta\right)\hat{c}_t - b\mu_z\hat{c}_{t-1} + b\mu_z\left(\hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right) + \\
+\left(\mu_z - b\beta\right)\left(\mu_z - b\right)\hat{\psi}_{z,t} + \frac{\tau^c}{1+\tau^c}\left(\mu_z - b\beta\right)\left(\mu_z - b\right)\hat{\tau}_t^c + \left(\mu_z - b\beta\right)\left(\mu_z - b\right)\hat{\gamma}_t^c \\
-\left(\mu_z - b\right)\left(\mu_z\hat{\zeta}_t^c - b\beta\hat{\zeta}_{t+1}^c\right)
\end{array}\right] = 0
$$

7.2 Log-linearizing the first-order condition w.r.t. m_{t+1}

The first-order condition for m_{t+1} is

$$-\upsilon_t + \beta \mathrm{E}_t \left[\upsilon_{t+1} R_t - \upsilon_{t+1} \tau^k_{t+1} \left(R_t - 1 \right) \right] = 0$$

which is equivalent to, using the definition $\psi_t \equiv \upsilon_t P_t$,

$$-\psi_t + \beta \mathrm{E}_t \left[\psi_{t+1} \frac{R_t}{\pi_{t+1}} - \frac{\psi_{t+1}}{\pi_{t+1}} \tau^k_{t+1} \left(R_t - 1 \right) \right] = 0.$$

Using functional forms in the first-order conditions and scaling with the technology level, we finally obtain the following first-order condition

$$-\psi_{z,t} + \beta \mathrm{E}_t \left[\frac{\psi_{z,t+1}}{\mu_{z,t+1}} \frac{R_t}{\pi_{t+1}} - \frac{1}{\mu_{z,t+1}} \frac{\psi_{z,t+1}}{\pi_{t+1}} \tau^k_{t+1} \left(R_t - 1 \right) \right] = 0 \tag{249}$$

Note that $\psi_{z,t}$ in steady state equals:

$$\psi_z = \frac{1}{c \left(1 + \tau^k \right) \gamma^c} \left(\frac{\mu_z - b\beta}{\mu_z - b} \right)$$

Now let's totally differentiate (249):

$$\begin{aligned} F \left(\psi_{z,t}, \psi_{z,t+1}, \mu_{z,t+1}, R_t, \pi_{t+1}, \tau^k_{t+1} \right) &= \\ -\psi_{z,t} + \beta \left(\frac{\psi_{z,t+1}}{\mu_{z,t+1}} \frac{R_t}{\pi_{t+1}} - \frac{1}{\mu_{z,t+1}} \frac{\psi_{z,t+1}}{\pi_{t+1}} \tau^k_{t+1} \left(R_t - 1 \right) \right) &= 0 \end{aligned}$$

$$\begin{aligned} F_1 \left(\psi_{z,t}, \psi_{z,t+1}, \mu_{z,t+1}, R_t, \pi_{t+1}, \tau^k_{t+1} \right) &= \\ &-1 \end{aligned}$$

$$
\begin{aligned}
& F_2\left(\psi_{z,t}, \psi_{z,t+1}, \mu_{z,t+1}, R_t, \pi_{t+1}, \tau^k_{t+1}\right) \\
= \quad & \beta\left(\frac{1}{\mu_{z,t+1}}\frac{R_t}{\pi_{t+1}} - \frac{1}{\mu_{z,t+1}}\frac{1}{\pi_{t+1}}\tau^y_{t+1}\left(R_t - 1\right)\right) \\
& \beta\left(\frac{1}{\mu_z}\frac{R}{\pi} - \frac{1}{\mu_z}\frac{1}{\pi}\tau^y\left(R-1\right)\right) \\
= \quad & -\beta\frac{-R + \tau^k R - \tau^k}{\mu_z \pi}
\end{aligned}
$$

$$
\begin{aligned}
& F_3\left(\psi_{z,t}, \psi_{z,t+1}, \mu_{z,t+1}, R_t, \pi_{t+1}, \tau^k_{t+1}\right) \\
= \quad & -\beta\mu_{z,t+1}^{-2}\left(\psi_{z,t+1}\frac{R_t}{\pi_{t+1}} - \frac{\psi_{z,t+1}}{\pi_{t+1}}\tau^y_{t+1}\left(R_t - 1\right)\right) \\
& -\beta\mu_z^{-2}\left(\psi_z\frac{R}{\pi} - \frac{\psi_z}{\pi}\tau^y\left(R-1\right)\right) \\
= \quad & \beta\psi_z\frac{-R + \tau^k R - \tau^k}{\mu_z^2 \pi}
\end{aligned}
$$

$$
\begin{aligned}
& F_4\left(\psi_{z,t}, \psi_{z,t+1}, \mu_{z,t+1}, R_t, \pi_{t+1}, \tau^k_{t+1}\right) \\
= \quad & \beta\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\frac{1}{\pi_{t+1}}\left(1 - \tau^k_{t+1}\right) \\
& \beta\frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(1 - \tau^k\right)
\end{aligned}
$$

$$
\begin{aligned}
& F_5\left(\psi_{z,t},\psi_{z,t+1},\mu_{z,t+1},R_t,\pi_{t+1},\tau^k_{t+1}\right) \\
= & -\beta\pi_{t+1}^{-2}\left(\frac{\psi_{z,t+1}}{\mu_{z,t+1}}R_t-\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\tau^k_{t+1}\left(R_t-1\right)\right) \\
= & -\frac{\beta}{\pi_{t+1}^2}\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\left(R_t-\tau^k_{t+1}R_t+\tau^k_{t+1}\right) \\
& -\frac{\beta}{\pi^2}\frac{\psi_z}{\mu_z}\left(R-\tau^kR+\tau^k\right) \\
= & \beta\psi_z\frac{-R+\tau^kR-\tau^k}{\pi^2\mu_z}
\end{aligned}
$$

$$
\begin{aligned}
& F_6\left(\psi_{z,t},\psi_{z,t+1},\mu_{z,t+1},R_t,\pi_{t+1},\tau^k_{t+1}\right) \\
= & \frac{\beta}{\mu_{z,t+1}}\frac{\psi_{z,t+1}}{\pi_{t+1}}\left(1-R_t\right) \\
& \frac{\beta}{\mu_z}\frac{\psi_z}{\pi}\left(1-R\right)
\end{aligned}
$$

Collect terms:

$$
\begin{aligned}
& -\psi_z\hat{\psi}_{z,t}-\beta\frac{-R+\tau^kR-\tau^k}{\mu_z\pi}\psi_z\hat{\psi}_{z,t+1}+\beta\psi_z\frac{-R+\tau^kR-\tau^k}{\mu_z^2\pi}\mu_z\hat{\mu}_{z,t+1} \\
& +\beta\frac{\psi_z}{\mu_z}\frac{1}{\pi}\left(1-\tau^k\right)R\hat{R}_t+\beta\psi_z\frac{-R+\tau^kR-\tau^k}{\pi^2\mu_z}\pi\hat{\pi}_{t+1}+\frac{\beta}{\mu_z}\frac{\psi_z}{\pi}\left(1-R\right)\tau^k\hat{\tau}^k_{t+1}
\end{aligned}
$$

Divide by ψ_z :

$$
\begin{aligned}
& -\hat{\psi}_{z,t}-\beta\frac{-R+\tau^kR-\tau^k}{\mu_z\pi}\hat{\psi}_{z,t+1}+\beta\frac{-R+\tau^kR-\tau^k}{\mu_z\pi}\hat{\mu}_{z,t+1} \\
& +\beta\frac{1}{\mu_z}\frac{1}{\pi}\left(1-\tau^k\right)R\hat{R}_t+\beta\frac{-R+\tau^kR-\tau^k}{\pi\mu_z}\hat{\pi}_{t+1}+\frac{\beta}{\mu_z}\frac{1}{\pi}\left(1-R\right)\tau^k\hat{\tau}^k_{t+1}
\end{aligned}
$$

Multiply by $\mu_z\pi$

$$-\mu_z \pi \hat{\psi}_{z,t} - \beta \left(-R + \tau^k R - \tau^k\right) \hat{\psi}_{z,t+1} + \beta \left(-R + \tau^k R - \tau^k\right) \hat{\mu}_{z,t+1}$$
$$+\beta \left(1 - \tau^k\right) R \hat{R}_t + \beta \left(-R + \tau^k R - \tau^k\right) \hat{\pi}_{t+1} + \beta \left(1 - R\right) \tau^k \hat{\tau}^k_{t+1}$$

Thus, in linearized form, (249) reduces to:

$$E_t \left[\begin{array}{c} -\mu_z \bar{\pi} \hat{\psi}_{z,t} + \beta \left(R - \tau^k R + \tau^k\right) \hat{\psi}_{z,t+1} - \beta \left(R - \tau^k R + \tau^k\right) \hat{\mu}_{z,t+1} \\ +\beta \left(1 - \tau^k\right) R \hat{R}_t - \beta \left(R - \tau^k R + \tau^k\right) \hat{\pi}_{t+1} + \beta \left(1 - R\right) \tau^k \hat{\tau}^k_{t+1} \end{array} \right] = 0 \qquad (-**-)$$

The stationary first order condition for m_{t+1} in steady-state equals:

$$\begin{aligned} -\psi_z + \beta \left(\frac{\psi_z}{\mu_z} \frac{R}{\pi} - \frac{1}{\mu_z} \frac{\psi_z}{\pi} \tau^k \left(R - 1\right) \right) &= 0 \\ &\Rightarrow R = \frac{-\mu_z \pi + \beta \tau^k}{\beta \left(-1 + \tau^k\right)} \\ &= \frac{\mu_z \pi - \beta \tau^k}{\beta \left(1 - \tau^k\right)} \end{aligned}$$

From the definitional equation for money growth, we have:

$$\begin{aligned} \pi &= \frac{\mu}{\mu_z} \\ R &= \frac{\mu - \beta \tau^k}{\beta \left(1 - \tau^k\right)} \end{aligned}$$

So, (I guess) $(- * * -)$ must equal

$$E_t \left[-\mu \hat{\psi}_{z,t} + \mu \hat{\psi}_{z,t+1} - \mu \hat{\mu}_{z,t+1} + \left(\mu - \beta \tau^k\right) \hat{R}_t - \mu \hat{\pi}_{t+1} + \frac{\tau^k}{1 - \tau^k} \left(\beta - \mu\right) \hat{\tau}^k_{t+1} \right] = 0$$

7.3 Log-linearizing the first-order condition w.r.t. $\bar{k}_{t+1}$

The first-order condition for $\bar{k}_{t+1}$ is

$$-\omega_t + \beta \mathrm{E}_t \left[(1-\delta)\omega_{t+1} + \upsilon_{t+1} \left(\left(1-\tau^k_{t+1}\right) R^k_{t+1} u_{t+1} - P_{t+1} a\left(u_{t+1}\right)\right)\right] = 0,$$

which can be rearranged as follows

$$-\frac{\omega_t}{\upsilon_t P_t}\upsilon_t P_t + \beta \mathrm{E}_t \left[(1-\delta)\frac{\omega_{t+1}}{\upsilon_{t+1}P_{t+1}}\upsilon_{t+1}P_{t+1} + \upsilon_{t+1}P_{t+1}\left(\left(1-\tau^k_{t+1}\right)\frac{R^k_{t+1}}{P_{t+1}}u_{t+1} - a\left(u_{t+1}\right)\right)\right] = 0,$$

and using $\psi_t \equiv \upsilon_t P_t$, $P_{k',t} = \frac{\omega_t}{\upsilon_t P_t}$ and $r^k_t \equiv \frac{R^k_t}{P_t}$, we obtain

$$-P_{k',t}\psi_t + \beta \mathrm{E}_t \left[(1-\delta)P_{k',t+1}\psi_{t+1} + \psi_{t+1}\left(\left(1-\tau^k_{t+1}\right) r^k_{t+1}u_{t+1} - a\left(u_{t+1}\right)\right)\right] = 0, \tag{250}$$

Stationarize (250):

$$-P_{k',t}\psi_{z,t} + \beta \mathrm{E}_t \left[\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\left((1-\delta)P_{k',t+1} + \left(\left(1-\tau^k_{t+1}\right) r^k_{t+1}u_{t+1} - a\left(u_{t+1}\right)\right)\right)\right] = 0$$

Totally differentiate (250) and evaluate in ss $\left(u_t = \frac{k_t}{\bar{k}_t} = u = 1 \Rightarrow a\left(u\right) = a\left(1\right) = 0\right)$:

$$\begin{aligned} & F\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau^k_{t+1}, r^k_{t+1}, u_{t+1}, \mu_{z,t+1}\right) \\ = \quad & -P_{k',t}\psi_{z,t} + \beta\left[\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\left((1-\delta)P_{k',t+1} + \left(\left(1-\tau^k_{t+1}\right) r^k_{t+1}u_{t+1} - a\left(u_{t+1}\right)\right)\right)\right] = 0 \end{aligned}$$

$$\begin{aligned} & F_1\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau^k_{t+1}, r^k_{t+1}, u_{t+1}, \mu_{z,t+1}\right) \\ = \quad & -\psi_{z,t} \\ ss \quad : \quad & -\psi_z \end{aligned}$$

$$
\begin{aligned}
& F_2\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau^k_{t+1}, r^k_{t+1}, u_{t+1}, \mu_{z,t+1}\right) \\
= \quad & \beta(1-\delta)\frac{\psi_{z,t+1}}{\mu_{z,t+1}} \\
ss \quad : \quad & \beta(1-\delta)\frac{\psi_z}{\mu_z}
\end{aligned}
$$

$$
\begin{aligned}
& F_3\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau^k_{t+1}, r^k_{t+1}, u_{t+1}, \mu_{z,t+1}\right) \\
= \quad & -P_{k',t} \\
ss \quad : \quad & -P_{k'}
\end{aligned}
$$

$$
\begin{aligned}
& F_4\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau^k_{t+1}, r^k_{t+1}, u_{t+1}, \mu_{z,t+1}\right) \\
= \quad & \beta\left[\frac{1}{\mu_{z,t+1}}\left((1-\delta)P_{k',t+1} + \left(\left(1-\tau^k_{t+1}\right) r^k_{t+1} u_{t+1} - a\left(u_{t+1}\right)\right)\right)\right] \\
ss \quad : \quad & \beta\left[\frac{1}{\mu_z}\left((1-\delta)P_{k'} + \left(\left(1-\tau^k\right) r^k\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& F_5\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau^k_{t+1}, r^k_{t+1}, u_{t+1}, \mu_{z,t+1}\right) \\
= \quad & -\frac{\psi_{z,t+1}}{\mu_{z,t+1}} r^k_{t+1} u_{t+1} \beta \\
ss \quad : \quad & -\frac{\psi_z}{\mu_z} r^k \beta
\end{aligned}
$$

$$
\begin{aligned}
& F_6\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau_{t+1}^k, r_{t+1}^k, u_{t+1}, \mu_{z,t+1}\right) \\
= \quad & \frac{\psi_{z,t+1}}{\mu_{z,t+1}} u_{t+1}\beta - \frac{\psi_{z,t+1}}{\mu_{z,t+1}} u_{t+1}\tau_{t+1}^k \beta \\
= \quad & \frac{\psi_{z,t+1}}{\mu_{z,t+1}} u_{t+1}\beta\left(1-\tau_{t+1}^k\right) \\
ss \quad : \quad & \frac{\psi_z}{\mu_z}\beta\left(1-\tau^k\right)
\end{aligned}
$$

$$
\begin{aligned}
& F_7\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau_{t+1}^k, r_{t+1}^k, u_{t+1}, \mu_{z,t+1}\right) \\
= \quad & \frac{\psi_{z,t+1}}{\mu_{z,t+1}} r_{t+1}^k \beta - \frac{\psi_{z,t+1}}{\mu_{z,t+1}} r_{t+1}^k \tau_{t+1}^k \beta - \frac{\psi_{z,t+1}}{\mu_{z,t+1}} \beta a'\left(u_{t+1}\right) \\
= \quad & \beta\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\left(\left(1-\tau_{t+1}^k\right) r_{t+1}^k - a'\left(u_{t+1}\right)\right) \\
ss \quad : \quad & \beta\frac{\psi_z}{\mu_z}\left(\left(1-\tau^k\right) r^k - \left(1-\tau^k\right) r^k\right) \\
= \quad & 0
\end{aligned}
$$

$$
\begin{aligned}
& F_8\left(P_{k',t}, P_{k',t+1}, \psi_{z,t}, \psi_{z,t+1}, \tau_{t+1}^k, r_{t+1}^k, u_{t+1}, \mu_{z,t+1}\right) \\
= \quad & -\beta\left[\frac{\psi_{z,t+1}}{\mu_{z,t+1}^2}\left((1-\delta)P_{k',t+1} + \left(\left(1-\tau_{t+1}^k\right) r_{t+1}^k u_{t+1} - a\left(u_{t+1}\right)\right)\right)\right] \\
ss \quad : \quad & -\beta\left[\frac{\psi_z}{\mu_z^2}\left((1-\delta)P_{k'} + \left(1-\tau^k\right) r^k\right)\right]
\end{aligned}
$$

Collect terms:

$$
\begin{aligned}
&-\psi_z P_{k'} \hat{P}_{k',t} + \beta(1-\delta)\frac{\psi_z}{\mu_z} P_{k'} \hat{P}_{k',t+1} - P_{k'} \psi_z \hat{\psi}_{z,t} \\
&+\beta \left[\frac{1}{\mu_z} \left((1-\delta) P_{k'} + \left(\left(1-\tau^k\right) r^k \right) \right) \right] \psi_z \hat{\psi}_{z,t+1} \\
&-\frac{\psi_z}{\mu_z} r^k \beta \tau^k \hat{\tau}^k_{t+1} + \frac{\psi_z}{\mu_z} \beta \left(1-\tau^k\right) r^k \hat{r}^k_{t+1} \\
&-\beta \frac{\psi_z}{\mu_z^2} \left[\left((1-\delta) P_{k'} + \left(1-\tau^k\right) r^k \right) \right] \mu_z \hat{\mu}_{z,t+1}
\end{aligned}
$$

Multiply with $\frac{\mu_z}{\psi_z}$:

$$
\begin{aligned}
&-\mu_z P_{k'} \hat{P}_{k',t} - \mu_z P_{k'} \hat{\psi}_{z,t} + \beta(1-\delta) P_{k'} \hat{P}_{k',t+1} \\
&+\beta \left[\left((1-\delta) P_{k'} + \left(\left(1-\tau^k\right) r^k \right) \right) \right] \hat{\psi}_{z,t+1} \\
&-r^k \beta \tau^k \hat{\tau}^k_{t+1} + \beta \left(1-\tau^k\right) r^k \hat{r}^k_{t+1} \\
&-\beta \left[(1-\delta) P_{k'} + \left(1-\tau^k\right) r^k \right] \hat{\mu}_{z,t+1}
\end{aligned}
$$

Make use of:

$$
r^k = \frac{\mu_z P_{k'} - \beta(1-\delta) P_{k'}}{\left(1-\tau^k\right)\beta}
$$

$$
\begin{aligned}
&-\mu_z P_{k'} \hat{P}_{k',t} - \mu_z P_{k'} \hat{\psi}_{z,t} + \beta(1-\delta) P_{k'} \hat{P}_{k',t+1} \\
&+\beta \left[\left((1-\delta) P_{k'} + \left(\left(1-\tau^y\right) \frac{\mu_z P_{k'} - \beta(1-\delta) P_{k'}}{\left(1-\tau^k\right)\beta} \right) \right) \right] \hat{\psi}_{z,t+1} \\
&-\frac{\mu_z P_{k'} - \beta(1-\delta) P_{k'}}{\left(1-\tau^k\right)\beta} \beta \tau^k \hat{\tau}^k_{t+1} + \beta \left(1-\tau^k\right) \frac{\mu_z P_{k'} - \beta(1-\delta) P_{k'}}{\left(1-\tau^k\right)\beta} \hat{r}^k_{t+1} \\
&-\beta \left[(1-\delta) P_{k'} + \left(1-\tau^k\right) \frac{\mu_z P_{k'} - \beta(1-\delta) P_{k'}}{\left(1-\tau^k\right)\beta} \right] \hat{\mu}_{z,t+1}
\end{aligned}
$$

or

$$-\mu_z P_{k'}\hat{P}_{k',t} - \mu_z P_{k'}\hat{\psi}_{z,t} + \beta(1-\delta)P_{k'}\hat{P}_{k',t+1} + \mu_z P_{k'}\hat{\psi}_{z,t+1}$$
$$-\frac{\mu_z - \beta(1-\delta)}{(1-\tau^k)\beta} P_{k'}\beta\tau^k\hat{\tau}^k_{t+1} + (\mu_z - \beta(1-\delta)) P_{k'}\hat{r}^k_{t+1} - \mu_z P_{k'}\hat{\mu}_{z,t+1}$$

Divide by $P_{k'}$:

$$-\mu_z \frac{P_{k'}}{P_{k'}}\hat{P}_{k',t} - \mu_z \frac{P_{k'}}{P_{k'}}\hat{\psi}_{z,t} + \beta(1-\delta)\frac{P_{k'}}{P_{k'}}\hat{P}_{k',t+1} + \mu_z \frac{P_{k'}}{P_{k'}}\hat{\psi}_{z,t+1}$$
$$-\frac{\mu_z - \beta(1-\delta)}{(1-\tau^k)\beta} \frac{P_{k'}}{P_{k'}}\beta\tau^k\hat{\tau}^k_{t+1} + (\mu_z - \beta(1-\delta)) \frac{P_{k'}}{P_{k'}}\hat{r}^k_{t+1} - \mu_z \frac{P_{k'}}{P_{k'}}\hat{\mu}_{z,t+1}$$

or:

$$-\mu_z\hat{P}_{k',t} - \mu_z\hat{\psi}_{z,t} + \beta(1-\delta)\hat{P}_{k',t+1} + \mu_z\hat{\psi}_{z,t+1}$$
$$-\frac{\mu_z - \beta(1-\delta)}{(1-\tau^k)\beta}\beta\tau^k\hat{\tau}^k_{t+1} + (\mu_z - \beta(1-\delta))\hat{r}^k_{t+1} - \mu_z\hat{\mu}_{z,t+1}$$

Divide by μ_z :

$$\hat{\psi}_{z,t} + \hat{\mu}_{z,t+1} - \hat{\psi}_{z,t+1} - \frac{\beta(1-\delta)}{\mu_z}\hat{P}_{k',t+1} + \hat{P}_{k',t} - \frac{\mu_z - \beta(1-\delta)}{\mu_z}\hat{r}^k_{t+1}$$
$$+\frac{\mu_z - \beta(1-\delta)}{(1-\tau^k)\beta\mu_z}\beta\tau^k\hat{\tau}^k_{t+1}$$

We end up with

$$E_t\left[\begin{array}{c}\hat{\psi}_{z,t} + \hat{\mu}_{z,t+1} - \hat{\psi}_{z,t+1} - \frac{\beta(1-\delta)}{\mu_z}\hat{P}_{k',t+1} + \hat{P}_{k',t} - \frac{\mu_z-\beta(1-\delta)}{\mu_z}\hat{r}^k_{t+1} \\ +\frac{\tau^k}{(1-\tau^k)}\frac{\mu_z-\beta(1-\delta)}{\mu_z}\hat{\tau}^k_{t+1}\end{array}\right] = 0$$

Compare with $ACEL$:

$$E_t \left\{ \hat{\psi}_{z,t} + \hat{\mu}_{z,t+1} - \hat{\psi}_{z,t+1} - \frac{\beta(1-\delta)}{\mu_z \Upsilon^{\frac{1}{1-\alpha}}} \hat{p}_{k',t+1} + \hat{p}_{k',t} - \frac{\mu_z \Upsilon^{\frac{1}{1-\alpha}} - \beta(1-\delta)}{\mu_z \Upsilon^{\frac{1}{1-\alpha}}} \hat{r}^k_{t+1} \right\} = 0$$

7.4 Log-linearizing the first-order condition w.r.t. i_t

First some preliminaries: We will adopt the specification by CEE and assume that

$$F(I_t, I_{t-1}) = (1 - S(I_t/I_{t-1})) I_t \tag{251}$$

where the functional form of S is given by

$$\begin{aligned} S(x) &= g_3 \left(e^{(g_1(x-\mu_z))} + \frac{g_1}{g_2} e^{(-g_2(x-\mu_z))} - \left(1 + \frac{g_1}{g_2}\right)\right) \qquad (252) \\ S'(x) &= g_1 g_3 \left(e^{(g_1(x-1))} - e^{(-g_2(x-1))} \right) \\ S'(\mu_z) &= 0 \\ s''(x) &= g_1 g_3 \left(g_1 e^{g_1(x-\mu_z)} + g_2 e^{-g_2(x-\mu_z)} \right) \\ s''(\mu_z) &= g_1 g_3 (g_1 + g_2) \qquad\qquad (= SS) \end{aligned}$$

g_1, g_2 and g_3 are all positive constants. Note that only the parameter S'' is identified and will be used in the model. Moreover, (251) and (252) implies

$$\begin{aligned} F_1(I_t, I_{t-1}) &\equiv \frac{\partial F(I_t, I_{t-1})}{\partial I_t} = -S'(I_t/I_{t-1}) I_t/I_{t-1} + (1 - S(I_t/I_{t-1})) \qquad (253) \\ F_2(I_t, I_{t-1}) &\equiv \frac{\partial F(I_t, I_{t-1})}{\partial I_{t-1}} = S'(I_t/I_{t-1}) \left(\frac{I_t}{I_{t-1}}\right)^2 \qquad (254) \end{aligned}$$

Stationarize (253) and (254):

$$\begin{aligned} F_1(i_t, i_{t-1}, \mu_{z,t}) &\equiv -S'\left(\frac{\frac{I_t}{z_t} z_t}{\frac{I_{t-1}}{z_{t-1}} z_{t-1}}\right) \frac{\frac{I_t}{z_t} z_t}{\frac{I_{t-1}}{z_{t-1}} z_{t-1}} + \left(1 - S\left(\frac{\frac{I_t}{z_t} z_t}{\frac{I_{t-1}}{z_{t-1}} z_{t-1}}\right)\right) \\ &= -S'\left(\frac{i_t \mu_{z,t}}{i_{t-1}}\right) \frac{i_t \mu_{z,t}}{i_{t-1}} + \left(1 - S\left(\frac{i_t \mu_{z,t}}{i_{t-1}}\right)\right) \\ F_2(I_t, I_{t-1}) &\equiv S'\left(\frac{i_t \mu_{z,t}}{i_{t-1}}\right) \left(\frac{i_t \mu_{z,t}}{i_{t-1}}\right)^2 \end{aligned}$$

Take second derivatives:

$$\begin{aligned}
F_1(i_t, i_{t-1}, \mu_{z,t}) &\equiv -S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t\mu_{z,t}}{i_{t-1}} + \left(1 - S(\frac{i_t\mu_{z,t}}{i_{t-1}})\right) \\
F_{11}(i_t, i_{t-1}, \mu_{z,t}) &= -S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{\mu_{z,t}}{i_{t-1}} - S^{''}(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t\mu_{z,t}}{i_{t-1}}\frac{\mu_{z,t}}{i_{t-1}} - S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{\mu_{z,t}}{i_{t-1}} \\
F_{12}(i_t, i_{t-1}, \mu_{z,t}) &= S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t\mu_{z,t}}{i_{t-1}^2} + S^{''}(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t\mu_{z,t}}{i_{t-1}}\frac{i_t\mu_{z,t}}{i_{t-1}^2} + S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t\mu_{z,t}}{i_{t-1}^2} \\
F_{13}(i_t, i_{t-1}, \mu_{z,t}) &= -S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t}{i_{t-1}} - S^{''}(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t\mu_{z,t}}{i_{t-1}}\frac{i_t}{i_{t-1}} - S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t}{i_{t-1}}
\end{aligned}$$

$$\begin{aligned}
F_2(I_t, I_{t-1}) &\equiv S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\left(\frac{i_t\mu_{z,t}}{i_{t-1}}\right)^2 \\
F_{21}(i_t, i_{t-1}, \mu_{z,t}) &= 2S'(\frac{i_t\mu_{z,t}}{i_{t-1}})i_t\left(\frac{\mu_{z,t}}{i_{t-1}}\right)^2 + S^{''}(\frac{i_t\mu_{z,t}}{i_{t-1}})\left(\frac{i_t\mu_{z,t}}{i_{t-1}}\right)^2\frac{\mu_{z,t}}{i_{t-1}} \\
F_{22}(i_t, i_{t-1}, \mu_{z,t}) &= -2S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\frac{i_t^2\mu_{z,t}^2}{i_{t-1}^3} - S^{''}(\frac{i_t\mu_{z,t}}{i_{t-1}})\left(\frac{i_t\mu_{z,t}}{i_{t-1}}\right)^2\frac{i_t\mu_{z,t}}{i_{t-1}^2} \\
F_{23}(i_t, i_{t-1}, \mu_{z,t}) &= 2S'(\frac{i_t\mu_{z,t}}{i_{t-1}})\mu_{z,t}\left(\frac{i_t}{i_{t-1}}\right)^2 + S^{''}(\frac{i_t\mu_{z,t}}{i_{t-1}})\left(\frac{i_t\mu_{z,t}}{i_{t-1}}\right)^2\frac{i_t}{i_{t-1}}
\end{aligned}$$

so that these foc and soc evaluated in ss $(i_t = i_{t-1}, S(\mu_z) = S'(\mu_z) = 0)$:

$$\begin{aligned}
F_1(i, i, \mu_z) &\equiv 1 \\
F_{11}(i_t, i_{t-1}, \mu_{z,t}) &= -S^{''}(\mu_z)\frac{\mu_z^2}{i} \\
F_{12}(i_t, i_{t-1}, \mu_{z,t}) &= S^{''}(\mu_z)\frac{\mu_z^2}{i} \\
F_{13}(i_t, i_{t-1}, \mu_{z,t}) &= -S^{''}(\mu_z)\mu_z
\end{aligned}$$

$$
\begin{aligned}
F_2(i,i,\mu_z) &\equiv 0 \\
F_{21}(i,i,\mu_z) &= S^{\prime\prime}(\mu_z)\frac{\mu_z^3}{i} \\
F_{22}(i,i,\mu_z) &= -S^{\prime\prime}(\mu_z)\frac{\mu_z^3}{i} \\
F_{23}(i,i,\mu_z) &= S^{\prime\prime}(\mu_z))\mu_z^2
\end{aligned}
$$

The first order condition with respect to i_t equals (after scaling with the technology level and using $\frac{P_t^i}{P_t} = \gamma_t^i$):

$$
-\psi_{z,t}\gamma_t^i + P_{k',t}\psi_{z,t}\Upsilon_t F_1(i_t, i_{t-1}, \mu_{z,t}) + \beta E_t\left[P_{k',t+1}\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\Upsilon_{t+1}F_2(i_{t+1}, i_t, \mu_{z,t+1})\right] = 0 \qquad (\#)
$$

Note that (#) equals:

$$
-\psi_z\gamma^i + P_{k'}\psi_z\Upsilon = 0
$$

in ss. This implies that:

$$
P_{k'} = \frac{\gamma^i}{\Upsilon}.
$$

Now, totally differentiate (#) and evaluate in ss :

$$
\begin{aligned}
& F\left(\psi_{z,t}, \psi_{z,t+1}, \gamma_t^i, P_{k',t}, P_{k',t+1}, \Upsilon_t, \Upsilon_{t+1}, i_{t+1}, i_t, i_{t-1}, \mu_{z,t}, \mu_{z,t+1}\right) \\
= \quad & -\psi_{z,t}\gamma_t^i + P_{k',t}\psi_{z,t}\Upsilon_t F_1(i_t, i_{t-1}, \mu_{z,t}) + \beta E_t\left[P_{k',t+1}\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\Upsilon_{t+1}F_2(i_{t+1}, i_t, \mu_{z,t+1})\right]
\end{aligned}
$$

$$
\begin{aligned}
& F_1\left(\psi_{z,t}, \psi_{z,t+1}, \gamma_t^i, P_{k',t}, P_{k',t+1}, \Upsilon_t, \Upsilon_{t+1}, i_{t+1}, i_t, i_{t-1}, \mu_{z,t}, \mu_{z,t+1}\right) \\
= \quad & -\gamma_t^i + P_{k',t}\Upsilon_t F_1(i_t, i_{t-1}, \mu_{z,t}) \\
ss \quad : \quad & -\gamma^i + P_{k'}\Upsilon
\end{aligned}
$$

$$
\begin{aligned}
& F_2\left(\psi_{z,t}, \psi_{z,t+1}, \gamma_t^i, P_{k',t}, P_{k',t+1}, \Upsilon_t, \Upsilon_{t+1}, i_{t+1}, i_t, i_{t-1}, \mu_{z,t}, \mu_{z,t+1}\right) \\
= \quad & \beta P_{k',t+1}\frac{1}{\mu_{z,t+1}}\Upsilon_{t+1} F_2(i_{t+1}, i_t, \mu_{z,t+1}) \\
ss \quad : \quad & 0
\end{aligned}
$$

$$
\begin{aligned}
& F_3\left(\psi_{z,t}, \psi_{z,t+1}, \gamma_t^i, P_{k',t}, P_{k',t+1}, \Upsilon_t, \Upsilon_{t+1}, i_{t+1}, i_t, i_{t-1}, \mu_{z,t}, \mu_{z,t+1}\right) \\
= \quad & -\psi_{z,t} \\
ss \quad : \quad & -\psi_z
\end{aligned}
$$

$$
\begin{aligned}
& F_4\left(\psi_{z,t}, \psi_{z,t+1}, \gamma_t^i, P_{k',t}, P_{k',t+1}, \Upsilon_t, \Upsilon_{t+1}, i_{t+1}, i_t, i_{t-1}, \mu_{z,t}, \mu_{z,t+1}\right) \\
= \quad & \psi_{z,t}\Upsilon_t F_1(i_t, i_{t-1}, \mu_{z,t}) \\
ss \quad : \quad & \psi_z\Upsilon
\end{aligned}
$$

$$
\begin{aligned}
& F_5\left(\psi_{z,t}, \psi_{z,t+1}, \gamma_t^i, P_{k',t}, P_{k',t+1}, \Upsilon_t, \Upsilon_{t+1}, i_{t+1}, i_t, i_{t-1}, \mu_{z,t}, \mu_{z,t+1}\right) \\
= \quad & \beta\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\Upsilon_{t+1} F_2(i_{t+1}, i_t, \mu_{z,t+1}) \\
ss \quad : \quad & 0
\end{aligned}
$$

$$
\begin{aligned}
& F_6\left(\psi_{z,t},\psi_{z,t+1},\gamma_t^i,P_{k',t},P_{k',t+1},\Upsilon_t,\Upsilon_{t+1},i_{t+1},i_t,i_{t-1},\mu_{z,t},\mu_{z,t+1}\right) \\
= \quad & P_{k',t}\psi_{z,t}F_1(i_t,i_{t-1},\mu_{z,t}) \\
ss \quad : \quad & P_{k'}\psi_z
\end{aligned}
$$

$$
\begin{aligned}
& F_7\left(\psi_{z,t},\psi_{z,t+1},\gamma_t^i,P_{k',t},P_{k',t+1},\Upsilon_t,\Upsilon_{t+1},i_{t+1},i_t,i_{t-1},\mu_{z,t},\mu_{z,t+1}\right) \\
= \quad & \beta P_{k',t+1}\frac{\psi_{z,t+1}}{\mu_{z,t+1}}F_2(i_{t+1},i_t,\mu_{z,t+1}) \\
ss \quad : \quad & 0
\end{aligned}
$$

$$
\begin{aligned}
& F_8\left(\psi_{z,t},\psi_{z,t+1},\gamma_t^i,P_{k',t},P_{k',t+1},\Upsilon_t,\Upsilon_{t+1},i_{t+1},i_t,i_{t-1},\mu_{z,t},\mu_{z,t+1}\right) \\
= \quad & \beta P_{k',t+1}\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\Upsilon_{t+1}F_{21}(i_{t+1},i_t,\mu_{z,t+1}) \\
ss \quad : \quad & \beta P_{k'}\frac{\psi_z}{\mu_z}\Upsilon S^{''}(\mu_z)\frac{\mu_z^3}{i}
\end{aligned}
$$

$$
\begin{aligned}
& F_9\left(\psi_{z,t},\psi_{z,t+1},\gamma_t^i,P_{k',t},P_{k',t+1},\Upsilon_t,\Upsilon_{t+1},i_{t+1},i_t,i_{t-1},\mu_{z,t},\mu_{z,t+1}\right) \\
= \quad & P_{k',t}\psi_{z,t}\Upsilon_tF_{11}(i_t,i_{t-1},\mu_{z,t})+\beta P_{k',t+1}\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\Upsilon_{t+1}F_{22}(i_{t+1},i_t,\mu_{z,t+1}) \\
ss \quad : \quad & -P_{k'}\psi_z\Upsilon S^{''}(\mu_z)\frac{\mu_z^2}{i}-\beta P_{k'}\frac{\psi_z}{\mu_z}\Upsilon S^{''}(\mu_z)\frac{\mu_z^3}{i}
\end{aligned}
$$

$$
\begin{aligned}
& F_{10}\left(\psi_{z,t},\psi_{z,t+1},\gamma_t^i,P_{k',t},P_{k',t+1},\Upsilon_t,\Upsilon_{t+1},i_{t+1},i_t,i_{t-1},\mu_{z,t},\mu_{z,t+1}\right) \\
= \quad & P_{k',t}\psi_{z,t}\Upsilon_tF_{12}(i_t,i_{t-1},\mu_{z,t}) \\
ss \quad : \quad & P_{k'}\psi_z\Upsilon S^{''}(\mu_z)\frac{\mu_z^2}{i}
\end{aligned}
$$

$$
\begin{aligned}
& F_{11}\left(\psi_{z,t},\psi_{z,t+1},\gamma_t^i,P_{k',t},P_{k',t+1},\Upsilon_t,\Upsilon_{t+1},i_{t+1},i_t,i_{t-1},\mu_{z,t},\mu_{z,t+1}\right) \\
= \; & P_{k',t}\psi_{z,t}\Upsilon_t F_{13}(i_t,i_{t-1},\mu_{z,t}) \\
ss \; : \; & -P_{k'}\psi_z\Upsilon S^{''}(\mu_z)\mu_z
\end{aligned}
$$

$$
\begin{aligned}
& F_{12}\left(\psi_{z,t},\psi_{z,t+1},\gamma_t^i,P_{k',t},P_{k',t+1},\Upsilon_t,\Upsilon_{t+1},i_{t+1},i_t,i_{t-1},\mu_{z,t},\mu_{z,t+1}\right) \\
= \; & \beta P_{k',t+1}\frac{\psi_{z,t+1}}{\mu_{z,t+1}}\Upsilon_{t+1}F_{23}(i_{t+1},i_t,\mu_{z,t+1}) \\
ss \; : \; & \beta P_{k'}\frac{\psi_z}{\mu_z}\Upsilon S^{''}(\mu_z)\mu_z^2
\end{aligned}
$$

Collecting terms:

$$
\begin{aligned}
& -\gamma^i\psi_z\hat{\psi}_{z,t}+P_{k'}\Upsilon\psi_z\hat{\psi}_{z,t}-\psi_z\gamma^i\hat{\gamma}_t^i+\psi_z\Upsilon P_{k'}\hat{P}_{k',t}+P_{k'}\psi_z\Upsilon\hat{\Upsilon}_t \\
& +\beta P_{k'}\frac{\psi_z}{\mu_z}\Upsilon S^{''}(\mu_z)\frac{\mu_z^3}{i}i\hat{\imath}_{t+1}+\left(-P_{k'}\psi_z\Upsilon S^{''}(\mu_z)\frac{\mu_z^2}{i}-\beta P_{k'}\frac{\psi_z}{\mu_z}\Upsilon S^{''}(\mu_z)\frac{\mu_z^3}{i}\right)i\hat{\imath}_t \\
& +P_{k'}\psi_z\Upsilon S^{''}(\mu_z)\frac{\mu_z^2}{i}i\hat{\imath}_{t-1}-P_{k'}\psi_z\Upsilon S^{''}(\mu_z)\mu_z\mu_z\hat{\mu}_{z,t}+\beta P_{k'}\frac{\psi_z}{\mu_z}\Upsilon S^{''}(\mu_z))\mu_z^2\mu_z\hat{\mu}_{z,t+1}
\end{aligned}
$$

Divide with ψ_z :

$$
\begin{aligned}
& -\gamma^i\hat{\psi}_{z,t}+P_{k'}\Upsilon\hat{\psi}_{z,t}-\gamma^i\hat{\gamma}_t^i+\Upsilon P_{k'}\hat{P}_{k',t}+P_{k'}\Upsilon\hat{\Upsilon}_t \\
& +\beta P_{k'}\frac{1}{\mu_z}\Upsilon S^{''}(\mu_z)\frac{\mu_z^3}{i}i\hat{\imath}_{t+1}-(1+\beta)\,\mu_z^2P_{k'}\Upsilon S^{''}(\mu_z)\hat{\imath}_t \\
& +P_{k'}\Upsilon S^{''}(\mu_z)\frac{\mu_z^2}{i}i\hat{\imath}_{t-1}-P_{k'}\Upsilon S^{''}(\mu_z)\mu_z\mu_z\hat{\mu}_{z,t}+\beta P_{k'}\frac{1}{\mu_z}\Upsilon S^{''}(\mu_z)\mu_z^2\mu_z\hat{\mu}_{z,t+1}
\end{aligned}
$$

or

$$-\gamma^i\hat{\psi}_{z,t}+P_{k'}\Upsilon\hat{\psi}_{z,t}-\gamma^i\hat{\gamma}_t^i+\Upsilon P_{k'}\hat{P}_{k',t}+P_{k'}\Upsilon\hat{\Upsilon}_t$$

$$\mu_z^2 P_{k'}\Upsilon S^{''}(\mu_z)\left[\hat{\imath}_{t-1}-(1+\beta)\hat{\imath}_t+\beta\hat{\imath}_{t+1}-\hat{\mu}_{z,t}+\beta\hat{\mu}_{z,t+1}\right]$$

or

$$\left(P_{k'}\Upsilon-\gamma^i\right)\hat{\psi}_{z,t}-\gamma^i\hat{\gamma}_t^i+\Upsilon P_{k'}\hat{P}_{k',t}+P_{k'}\Upsilon\hat{\Upsilon}_t$$

$$-\mu_z^2 P_{k'}\Upsilon S^{''}(\mu_z)\left[(\hat{\imath}_t-\hat{\imath}_{t-1})-\beta(\hat{\imath}_{t+1}-\hat{\imath}_t)+\hat{\mu}_{z,t}-\beta\hat{\mu}_{z,t+1}\right]$$

Use $P_{k'}=\frac{\gamma^i}{\Upsilon}=\gamma^i$

$$\left(\frac{\gamma^i}{\Upsilon}\Upsilon-\gamma^i\right)\hat{\psi}_{z,t}-\gamma^i\hat{\gamma}_t^i+\Upsilon\frac{\gamma^i}{\Upsilon}\hat{P}_{k',t}+\frac{\gamma^i}{\Upsilon}\Upsilon\hat{\Upsilon}_t$$

$$-\mu_z^2\frac{\gamma^i}{\Upsilon}\Upsilon S^{''}(\mu_z)\left[(\hat{\imath}_t-\hat{\imath}_{t-1})-\beta(\hat{\imath}_{t+1}-\hat{\imath}_t)+\hat{\mu}_{z,t}-\beta\hat{\mu}_{z,t+1}\right]$$

or

$$-\gamma^i\hat{\gamma}_t^i+\gamma^i\hat{P}_{k',t}+\gamma^i\hat{\Upsilon}_t$$

$$-\mu_z^2\gamma^i S^{''}(\mu_z)\left[(\hat{\imath}_t-\hat{\imath}_{t-1})-\beta(\hat{\imath}_{t+1}-\hat{\imath}_t)+\hat{\mu}_{z,t}-\beta\hat{\mu}_{z,t+1}\right]$$

Divide with γ^i

$$-\hat{\gamma}_t^i+\hat{P}_{k',t}+\hat{\Upsilon}_t-\mu_z^2 S^{''}(\mu_z)\left[(\hat{\imath}_t-\hat{\imath}_{t-1})-\beta(\hat{\imath}_{t+1}-\hat{\imath}_t)+\hat{\mu}_{z,t}-\beta\hat{\mu}_{z,t+1}\right]$$

Re-apply the expectations operator:

$$E_t \left\{ \hat{P}_{k',t} + \hat{\Upsilon}_t - \hat{\gamma}_t^i - \mu_z^2 S^{''}\left(\mu_z\right)\left[\left(\hat{\imath}_t - \hat{\imath}_{t-1}\right) - \beta\left(\hat{\imath}_{t+1} - \hat{\imath}_t\right) + \hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right] \right\} = 0$$

$$\hat{P}_{k',t} + \hat{\Upsilon}_t - \hat{\gamma}_t^i - \mu_z^2 S^{''}\left(\mu_z\right)\left[-\hat{\imath}_{t-1} - \beta\hat{\imath}_{t+1} + \hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right] = \mu_z^2 S^{''}\left(\mu_z\right)\left(1+\beta\right)\hat{\imath}_t$$

$$\hat{P}_{k',t} + \hat{\Upsilon}_t - \hat{\gamma}_t^i + \mu_z^2 S^{''}\left(\mu_z\right)\hat{\imath}_{t-1} + \mu_z^2 S^{''}\left(\mu_z\right)\beta\hat{\imath}_{t+1} - \mu_z^2 S^{''}\left(\mu_z\right)\hat{\mu}_{z,t} + \mu_z^2 S^{''}\left(\mu_z\right)\beta\hat{\mu}_{z,t+1} = \mu_z^2 S^{''}\left(\mu_z\right)\left(1+\beta\right)\hat{\imath}_t$$

$$\begin{aligned} \mu_z^2 S^{''}\left(\mu_z\right)\left(1+\beta\right)\hat{\imath}_t &= \hat{P}_{k',t} + \hat{\Upsilon}_t - \hat{\gamma}_t^i + \mu_z^2 S^{''}\left(\mu_z\right)\left[\hat{\imath}_{t-1} + \beta\hat{\imath}_{t+1} - \hat{\mu}_{z,t} + \beta\hat{\mu}_{z,t+1}\right] \\ \hat{\imath}_t &= \frac{1}{\mu_z^2 S''\left(\mu_z\right)\left(1+\beta\right)}\left(\hat{P}_{k',t} + \hat{\Upsilon}_t - \hat{\gamma}_t^i + \mu_z^2 S^{''}\left(\mu_z\right)\hat{\imath}_{t-1} + \mu_z^2 S^{''}\left(\mu_z\right)\beta\hat{\imath}_{t+1} - \mu_z^2 S^{''}\left(\mu_z\right)\hat{\mu}_{z} \right. \end{aligned}$$

Compare with $ACEL:$

$$E_t \left\{ \hat{P}_{k',t} + \hat{\mu}_{\Upsilon,t} - \left(\mu_z \Upsilon^{\frac{\alpha}{1-\alpha}}\right)^2 S^{''}\left(\mu_z \Upsilon^{\frac{\alpha}{1-\alpha}}\right)\left[\left(\hat{\imath}_t - \hat{\imath}_{t-1}\right) - \beta\left(\hat{\imath}_{t+1} - \hat{\imath}_t\right) + \hat{\mu}_{z,t} - \beta\hat{\mu}_{z,t+1}\right] \right\} = 0$$

8 Household Wage-Setting

The household is a monopoly supplier of its own (differentiated) labor. It sells labor to a firm which transforms household labor into a homogeneous input good, X. That producer's production function is:

$$X = \left[\int_0^1 (h_j)^{\frac{1}{\lambda_w}} \, dj \right]^{\lambda_w}, \; 1 \leq \lambda_w < \infty. \tag{255}$$

The producer's problem is to maximize (255) subject to:

$$W_t X_t - \int_0^1 W_{j,t} h_{j,t} dh,$$

where W_t is the price of X_t. The first order condition associated with this problem, which is the household's demand for labor curve, is:

$$h_{j,t+l} = \left[\frac{\tilde{W}_{j,t+l}}{W_{t+l}} \right]^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}. \tag{256}$$

8.1 Wage Equation

The wage rate set by the household that gets to reoptimize today is $\tilde{W}_{j,t}$. The household takes into account that if it does not get to reoptimize next period, it's wage rate then is

$$W_{j,t+1} = \pi_t \mu_{z,t+1} \tilde{W}_{j,t},$$

where

$$\mu_{z,t+1} = \frac{z_{t+1}}{z_t}.$$

In period $t+2$ it is

$$W_{j,t+2} = \pi_t \pi_{t+1} \mu_{z,t+1} \mu_{z,t+2} \tilde{W}_{j,t}$$

and

$$
\begin{aligned}
W_{j,t+l} &= \pi_t \times \cdots \times \pi_{t+l-1}\mu_{z,t+1} \times \cdots \times \mu_{z,t+l}\tilde{W}_{j,t}. \\
&= \frac{P_t}{P_{t-1}}\frac{P_{t+1}}{P_t} \times \cdots \times \frac{P_{t+l-1}}{P_{t+l}}\frac{z_{t+1}}{z_t}\frac{z_{t+2}}{z_{t+1}} \times \cdots \times \frac{z_{t+l}}{z_{t+l-1}}\tilde{W}_{j,t} \\
&= \frac{P_{t+l-1}}{P_{t-1}}\frac{z_{t+l}}{z_t}\tilde{W}_{j,t}
\end{aligned}
$$

[Technically, it is useful to note the slight difference in timing between inflation and the technology shock. The former reflects that indexing is lagged. The latter reflects that indexing to the technology shock is contemporaneous.]

Rewrite the demand that the individual household faces in the following way:

$$
\begin{aligned}
h_{j,t+l} &= \left(\frac{\tilde{W}_{j,t+l}}{W_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&= \left(\frac{\tilde{W}_{j,t+l}}{\frac{W_{t+l}}{z_{t+l}P_{t+l}} z_{t+l}P_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&\Rightarrow \{P_{t+l} = P_t\pi_t \times \cdots \times \pi_{t+l}\} \Rightarrow \\
&= \left(\frac{\tilde{W}_{j,t+l}}{\frac{W_{t+l}}{z_{t+l}P_{t+l}} z_{t+l}P_t\pi_t \times \cdots \times \pi_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&= \left(\frac{\tilde{W}_{j,t+l}}{w_{t+l}z_{t+l}P_t}\frac{1}{\pi_t \times \cdots \times \pi_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&\Rightarrow \left\{W_{j,t+l} = \pi_t \times \cdots \times \pi_{t+l-1}\mu_{z,t+1} \times \cdots \times \mu_{z,t+l}\tilde{W}_t\right\} \Rightarrow \\
&= \left(\frac{\tilde{W}_{j,t}\mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l}z_{t+l}P_t}X_{t,j}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}
\end{aligned}
$$

where $\tilde{W}_t$ denotes the nominal wage set by households that reoptimize in period t, and W_t denotes the nominal wage rate associated with aggregate, homogeneous labor, X_t. Also,

$$X_{t,l} = \frac{\pi_t \times \pi_{t+1} \times \cdots \times \pi_{t+l-1}}{\pi_{t+1} \times \cdots \times \pi_{t+l}} = \frac{\pi_t}{\pi_{t+l}}.$$

The j^{th} household that reoptimizes its wage, $\tilde{W}_{j,t}$, does so to optimize:

$$E_0^j \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \left[u\left(c_t^j - bC_{t-1}\right) + \zeta_t^h z\left(h_t^j\right) + \zeta_t^h V\left(\frac{q^j}{z_t P_t}\right)\right] \\ +\upsilon_t \left[\begin{array}{c} R_t\left(m_t - q_t\right) + q_t + \left(1-\tau_t^y\right)\Pi_t + \left(1-\tau_t^y\right)\frac{W_{j,t}}{1+\tau_t^w} h_{j,t} \\ +\left(1-\tau_t^y\right) R_t^k u_t \bar{k}_t + R_t^* \Phi\left(A_t\right) S_t b_t^* \\ -\tau_t^y\left[\left(R-1\right)\left(m_t - q_t\right) + \left(R_t^* - 1\right)\Phi\left(A_t\right) S_t b_t^*\right] + TR_t \\ -\left(m_{t+1} + S_t b_{t+1}^* + P_t^c c_t\left(1+\tau_t^c\right) + P_t^i i_t + P_t\left(a\left(u_t\right)\bar{k}_t + P_{k',t}\Delta_t\right)\right) \end{array} \right] \\ +\omega_t\left[\left(1-\delta\right)\bar{k}_t + \mu_{\Upsilon,t} F\left(i_t, i_{t-1}\right) + \Delta_t - \bar{k}_{t+1}\right] \end{array} \right]$$

Neglect irrelevant terms:

$$\begin{aligned} CEE/ACEL &: E_t \sum_{l=0}^{\infty} \left(\beta\xi_w\right)^{l-t} \{-\zeta_{t+l} z(h_{j,t+l}) + \upsilon_{t+l} W_{j,t+l} h_{j,t+l}\}, \\ RB &: E_t \sum_{l=0}^{\infty} \left(\beta\xi_w\right)^{l-t} \{-\zeta_{t+l} z(h_{j,t+l}) + \upsilon_{t+l} \frac{\left(1-\tau_{t+l}^y\right)}{\left(1+\tau_{t+l}^w\right)} W_{j,t+l} h_{j,t+l}\}, \end{aligned}$$

where

$$z(h) = \frac{(h)^{1+\sigma_L}}{1+\sigma_L} A_L$$

The presence of ξ_w by the discount factor reflects that the household is only concerned with the future states of the world in which it cannot reoptimize its wage.

Substituting out for $h_{j,t+l}$ using the demand curve:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(h_{j,t+l}) + \upsilon_{t+l} \frac{(1-\tau_{t+l}^y)}{(1+\tau_{t+l}^w)} h_{j,t+l}\}$$

$$\Rightarrow \left\{ h_{j,t+l} = \left(\frac{\tilde{W}_{t+l}}{W_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} = \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \right\} \Rightarrow$$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l})$$

$$+\upsilon_{t+l} \frac{(1-\tau_{t+l}^y)}{(1+\tau_{t+l}^w)} W_{j,t+l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}\}.$$

Now, make use of: $\psi_{t+l} = \upsilon_{t+l} \frac{(1-\tau_{t+l}^y)}{(1+\tau_{t+l}^w)} P_{t+l} \Rightarrow \upsilon_{t+l} \frac{(1-\tau_{t+l}^y)}{(1+\tau_{t+l}^w)} = \frac{\psi_{t+l}}{P_{t+l}}$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l})$$

$$+\frac{\psi_{t+l}}{P_{t+l}} W_{j,t+l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}\}.$$

Make use of the fact that:

$$\begin{aligned}
W_{j,t+l} &= \pi_t \times \cdots \times \pi_{t+l-1} \mu_{z,t+1} \times \cdots \times \mu_{z,t+l} \tilde{W}_t \\
P_{t+l} &= P_t \pi_{t+1} \times \cdots \times \pi_{t+l} = P_t \frac{P_{t+1}}{P_t} \frac{P_{t+2}}{P_{t+1}} \frac{P_{t+3}}{P_{t+2}} \times \cdots \times \frac{P_{t+l-1}}{P_{t+l-2}} \frac{P_{t+l}}{P_{t+l-1}} = P_t \frac{P_{t+l}}{P_t} = P_{t+l} \\
\frac{W_{j,t+l}}{P_{t+l}} &= \frac{\pi_t \times \cdots \times \pi_{t+l-1} \mu_{z,t+1} \times \cdots \times \mu_{z,t+l} \tilde{W}_t}{P_t \pi_{t+1} \times \cdots \times \pi_{t+l}} \\
&= \frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} \frac{\pi_t \times \cdots \times \pi_{t+l-1}}{\pi_{t+1} \times \cdots \times \pi_{t+l}} \\
\frac{W_{j,t+l}}{P_{t+l}} &= \frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} X_{t,l}
\end{aligned}$$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l})$$

$$+\psi_{t+l} \frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} X_{t,l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}\},$$

or, after rearranging:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l})$$

$$+\psi_{t+l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} \right) \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} \left(\frac{1}{z_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}\},$$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \{-\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l})$$

$$+\psi_{t+l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} \right)^{1+\frac{\lambda_w}{1-\lambda_w}} \left(\frac{1}{z_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}\},$$

or, $(\psi_{z,t+l} = z_{t+l}\psi_{t+l})$:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \left\{ \begin{array}{c} -\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \\ +z_{t+l}\psi_{t+l} \frac{1}{z_{t+l}} \left(\frac{1}{z_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} \right)^{1+\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \end{array} \right\}.$$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \left\{ \begin{array}{c} -\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\ +\psi_{z,t+l} \left(\frac{1}{z_{t+l}} \right)^{1+\frac{\lambda_w}{1-\lambda_w}} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{P_t} \right)^{1+\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \end{array} \right\}$$

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^{l-t} \left\{ \begin{array}{c} -\zeta_{t+l} z(\left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{w_{t+l} z_{t+l} P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \\ +\psi_{z,t+l} \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{z_{t+l} P_t} \right)^{1+\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \end{array} \right\}$$

But, note that:

$$\begin{aligned} z_{t+1} &= \mu_{z,t+1} z_t \\ z_{t+2} &= \mu_{z,t+2}\mu_{z,t+1} z_t, \end{aligned}$$

etc., so that

$$\frac{\mu_{z,t+1}\times\cdots\times\mu_{z,t+l}}{z_{t+l}} = \frac{\mu_{z,t+1}\times\cdots\times\mu_{z,t+l}}{\mu_{z,t+1}\times\cdots\times\mu_{z,t+l} z_t} = \frac{1}{z_t}.$$

Then,

$$E_t\sum_{l=0}^{\infty}(\beta\xi_w)^l\left\{-\zeta_{t+l}z(\left(\frac{\tilde{W}_t}{w_{t+l}z_tP_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l})+\psi_{z,t+l}\left(\frac{\tilde{W}_t}{z_tP_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l}\right\}.$$

$$(\#)$$

Differentiate $(\#)$ with respect to $\tilde{W}_t$:

$$\begin{aligned} & \frac{\partial\left(\left(\frac{\tilde{W}_t}{w_{t+l}z_tP_t}X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l}\right)}{\partial\tilde{W}_t} \\ = & \left(-W_t^{-\frac{2\lambda_w-1}{-1+\lambda_w}}w_{t+l}^{\frac{\lambda_w}{-1+\lambda_w}}z_t^{\frac{\lambda_w}{-1+\lambda_w}}P_t^{\frac{\lambda_w}{-1+\lambda_w}}X_{t,l}^{-\frac{\lambda_w}{-1+\lambda_w}}\frac{\lambda_w}{-1+\lambda_w}X_{t+l}\right) \\ = & \frac{\lambda_w}{1-\lambda_w}\tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}-1}\left(\frac{X_{t,l}}{w_{t+l}z_tP_t}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l} \\ & \frac{\partial\left(\psi_{z,t+l}\left(\frac{\tilde{W}_t}{z_tP_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l}\right)}{\partial\tilde{W}_t} \\ = & -\psi_{z,t+l}W_t^{-\frac{\lambda_w}{-1+\lambda_w}}z_t^{\frac{1}{-1+\lambda_w}}\frac{P_t^{\frac{1}{-1+\lambda_w}}}{-1+\lambda_w}X_{t,l}^{-\frac{1}{-1+\lambda_w}}w_{t+l}^{\frac{\lambda_w}{-1+\lambda_w}}X_{t+l} \\ = & \left(\psi_{z,t+l}\left(\frac{1}{1-\lambda_w}\right)\tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}}\left(\frac{1}{z_tP_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}}X_{t,l}\left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}}X_{t+l}\right) \end{aligned}$$

We now have the foc:

$$E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left\{ \begin{array}{c} -\zeta_{t+l} z'(\left(\frac{\tilde{W}_t}{w_{t+l} z_t P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \frac{\lambda_w}{1-\lambda_w} \tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}-1} \left(\frac{X_{t,l}}{w_{t+l} z_t P_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\ +\psi_{z,t+l} \left(\frac{1}{1-\lambda_w}\right) \tilde{W}_t^{\frac{\lambda_w}{1-\lambda_w}} \left(\frac{1}{z_t P_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \end{array} \right\} = 0.$$

Multiply by $\tilde{W}_t^{-\frac{\lambda_w}{1-\lambda_w}+1}(1-\lambda_w)/\lambda_w$:

$$0 = E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left\{ \begin{array}{c} -\zeta_{t+l} z'(\left(\frac{\tilde{W}_t}{w_{t+l} z_t P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \left(\frac{X_{t,l}}{w_{t+l} z_t P_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\ +\psi_{z,t+l} \left(\frac{1}{\lambda_w}\right) \tilde{W}_t \left(\frac{1}{z_t P_t}\right)^{1+\frac{\lambda_w}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \end{array} \right\}$$

Multiply by $P_t^{\frac{\lambda_w}{1-\lambda_w}}$:

$$0 = E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left\{ \begin{array}{c} -\zeta_{t+l} z'(\left(\frac{\tilde{W}_t}{w_{t+l} z_t P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \left(\frac{X_{t,l}}{w_{t+l} z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\ +\psi_{z,t+l} \frac{1}{\lambda_w} \frac{\tilde{W}_t}{P_t} \left(\frac{1}{z_t}\right)^{\frac{1}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \end{array} \right\}$$

Now get this in terms of stationary variables using $\tilde{w}_t = \tilde{W}_t / W_t$, $w_t = W_t / (z_t P_t)$:

$$0 = E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left\{ \begin{array}{c} -\zeta_{t+l} z'(\left(\frac{\tilde{w}_t W_t}{w_{t+l} z_t P_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \left(\frac{X_{t,l}}{w_{t+l} z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\ +\psi_{z,t+l} \frac{1}{\lambda_w} \frac{\tilde{w}_t W_t}{P_t} \left(\frac{1}{z_t}\right)^{\frac{1}{1-\lambda_w}} X_{t,l} \left(\frac{X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \end{array} \right\}$$

and

$$0 = E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left\{ \begin{array}{c} -\zeta_{t+l} z'(\left(\frac{\tilde{w}_t z_t w_t}{w_{t+l} z_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \left[\left(\frac{X_{t,l}}{w_{t+l} z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}\right] \\ +\psi_{z,t+l} \frac{1}{\lambda_w} \tilde{w}_t z_t w_t z_t^{\frac{\lambda_w}{1-\lambda_w}-\frac{1}{1-\lambda_w}} X_{t,l} \left[\left(\frac{X_{t,l}}{z_t w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}\right] \end{array} \right\}.$$

or,

$$
\begin{aligned}
0 &= E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{X_{t,l}}{w_{t+l}z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \left\{ \begin{array}{c} -\zeta_{t+l} z'((\left(\frac{\tilde{w}_t z_t w_t}{w_{t+l} z_t} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \\ +\psi_{z,t+l} \frac{1}{\lambda_w} \tilde{w}_t z_t w_t z_t^{\frac{\lambda_w}{1-\lambda_w} - \frac{1}{1-\lambda_w}} X_{t,l} \end{array} \right\} \\
z_t^{\frac{\lambda_w}{1-\lambda_w} - \frac{1}{1-\lambda_w}} &= \frac{1}{z_t} \\
0 &= E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{X_{t,l}}{w_{t+l}z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \left\{ \begin{array}{c} -\zeta_{t+l} z'((\left(\frac{\tilde{w}_t w_t}{w_{t+l}} X_{t,l}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l}) \\ +\psi_{z,t+l} \frac{1}{\lambda_w} \tilde{w}_t w_t X_{t,l} \end{array} \right\}
\end{aligned}
$$

or,

$$
0 = E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{X_{t,l}}{w_{t+l}z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \psi_{z,t+l} \frac{1}{\lambda_w} \left\{ \tilde{w}_t w_t X_{t,l} - \lambda_w \zeta_{t+l} \frac{z'_{t+l}}{\psi_{z,t+l}} \right\}.
$$

Finally, multiply both sides of this expression by

$$
(\tilde{w}_t w_t z_t)^{\frac{\lambda_w}{1-\lambda_w}},
$$

$$
\begin{aligned}
E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{X_{t,l}}{w_{t+l}z_t}\right)_{t+l}^{\frac{\lambda_w}{1-\lambda_w}} X \psi_{z,t+l} \frac{1}{\lambda_w} \left[(\tilde{w}_t w_t z_t)^{\frac{\lambda_w}{1-\lambda_w}} \right] \left\{ \tilde{w}_t w_t X_{t,l} - \lambda_w \zeta_{t+l} \frac{z'_{t+l}}{\psi_{z,t+l}} \right\} &= 0. \\
E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{\tilde{w}_t w_t z_t X_{t,l}}{w_{t+l}z_t}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \psi_{z,t+l} \frac{1}{\lambda_w} \left\{ \tilde{w}_t w_t X_{t,l} - \lambda_w \zeta_{t+l} \frac{z'_{t+l}}{\psi_{z,t+l}} \right\} &= 0. \\
E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{\tilde{w}_t w_t X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \frac{\psi_{z,t+l}}{\lambda_w} \left\{ \tilde{w}_t w_t X_{t,l} - \lambda_w \zeta_{t+l} \frac{z'_{t+l}}{\psi_{z,t+l}} \right\} &= 0.
\end{aligned}
$$

so that

$$
E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{\tilde{w}_t w_t X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \psi_{z,t+l} \frac{1}{\lambda_w} \left\{ \tilde{w}_t w_t X_{t,l} - \lambda_w \zeta_{t+l} \frac{z'_{t+l}}{\psi_{z,t+l}} \right\} = 0,
$$

or

$$
E_t \sum_{l=0}^{\infty} (\beta\xi_w)^l \left(\frac{\tilde{w}_t w_t X_{t,l}}{w_{t+l}}\right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \left\{ \frac{\psi_{z,t+l}}{\lambda_w} \tilde{w}_t w_t X_{t,l} - \zeta_{t+l} z'_{t+l} \right\} = 0,
$$

and

$$
\begin{aligned}
h_{t+l} &= \left(\frac{\tilde{w}_t w_t}{w_{t+l}} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&= \left(\frac{\frac{\tilde{W}_t}{W_t} \frac{W_t}{z_t P_t}}{\frac{W_{t+l}}{z_{t+l} P_{t+l}}} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&= \left(\frac{\tilde{W}_t}{\frac{W_{t+l}}{z_{t+l} P_{t+l}} z_t P_t} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&\Rightarrow \left\{ \frac{1}{z_t} = \frac{\mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{z_{t+l}} \right\} \Rightarrow \\
&= \left(\frac{\tilde{W}_t \mu_{z,t+1} \times \cdots \times \mu_{z,t+l}}{\frac{W_{t+l}}{z_{t+l} P_{t+l}} z_{t+l} P_t} X_{t,j} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
&= \left(\frac{\tilde{W}_{t+l}}{W_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \\
0 &= E_t \sum_{l=0}^{\infty} (\beta \xi_w)^l h_{t+l} \left\{ \frac{\psi_{z,t+l}}{\lambda_w} \tilde{w}_t w_t X_{t,l} - \zeta_{t+l} z'_{t+l} \right\}, \\
&\Rightarrow \left\{ \psi_{z,t+l} = z_{t+l} \upsilon_{t+l} \frac{(1-\tau^y_{t+l})}{(1+\tau^w_{t+l})} P_{t+l} \right\} \Rightarrow \\
0 &= E_t \sum_{l=0}^{\infty} (\beta \xi_w)^l h_{t+l} \{ z_{t+l} \upsilon_{t+l} \frac{(1-\tau^y_{t+l})}{(1+\tau^w_{t+l})} P_{t+l} \frac{1}{\lambda_w} \tilde{w}_t w_t X_{t,l} - \zeta_{t+l} z'_{t+l} \}
\end{aligned}
$$

using the demand curve. Rewriting this

$$
z'_{t+l} = z'(h) = \frac{\partial \left(A_L \frac{(h)^{1+\sigma_L}}{1+\sigma_L} \right)}{\partial h} = h^{\sigma_L} A_L
$$

$$
E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l h_{j,t+l} \left[\tilde{w}_t w_t \frac{\psi_{z,t+l}}{\lambda_w} X_{t,l} + \zeta_{t+l} f_L(h_{j,t+l}) \right] = 0,
$$

where

$$
f_L(h_{j,t+l}) = -z'_{t+l} = -A_L h^{\sigma_L}_{j,t+l}
$$

or,

$$E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l h_{j,t+l} \left[\tilde{w}_t w_t \frac{\psi_{z,t+l}}{\lambda_w} X_{t,l} + \zeta_{t+l} f_L \left(\left(\frac{\tilde{w}_t w_t X_{t,l}}{w_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \right) \right] = 0,$$

where

$$\begin{aligned}
\psi_{z,t+l} &= z_{t+l} \upsilon_{t+l} \frac{\left(1-\tau_{t+l}^y\right)}{\left(1+\tau_{t+l}^w\right)} \\
\psi_{z,t+l}^{\tau} &= z_{t+l} \upsilon_{t+l} \\
\psi_{z,t+l} &= \psi_{z,t+l}^{\tau} \frac{\left(1-\tau_{t+l}^y\right)}{\left(1+\tau_{t+l}^w\right)}
\end{aligned}$$

Writing this out, term by term:

$$\begin{aligned}
0 &= E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l h_{j,t+l} \left[\tilde{w}_t w_t \frac{\psi_{z,t+l}}{\lambda_w} X_{t,l} + \zeta_{t+l} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+l}} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \right) \right] = \\
& h_{j,t} \left[\tilde{w}_t w_t \frac{\psi_{z,t}}{\lambda_w} X_{t,0} + \zeta_t f_L \left((\tilde{w}_t X_{t,0})^{\frac{\lambda_w}{1-\lambda_w}} X_t \right) \right] \\
& + (\xi_w \beta)^1 h_{j,t+1} \left[\tilde{w}_t w_t \frac{\psi_{z,t+1}}{\lambda_w} X_{t,1} + \zeta_{t+1} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+1}} X_{t,1} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+1} \right) \right] \\
& + (\xi_w \beta)^2 h_{j,t+2} \left[\tilde{w}_t w_t \frac{\psi_{z,t+2}}{\lambda_w} X_{t,2} + \zeta_{t+2} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+2}} X_{t,2} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+2} \right) \right] \\
& + (\xi_w \beta)^3 h_{j,t+3} \left[\tilde{w}_t w_t \frac{\psi_{z,t+3}}{\lambda_w} X_{t,3} + \zeta_{t+3} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+3}} X_{t,3} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+3} \right) \right] \\
+..... &= 0
\end{aligned}$$

or

$$
\begin{aligned}
F &= 0 = E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l \, h_{j,t+l} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+l}^y\right)}{\left(1+\tau_{t+l}^w\right)} \frac{\psi_{z,t+l}^{\tau}}{\lambda_w} X_{t,l} + \zeta_{t+l} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+l}} X_{t,l} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \right) \right] = \\
& h_{j,t} \left[\tilde{w}_t w_t \frac{(1-\tau_t^y)}{(1+\tau_t^w)} \frac{\psi_{z,t}^{\tau}}{\lambda_w} X_{t,0} + \zeta_t f_L \left((\tilde{w}_t X_{t,0})^{\frac{\lambda_w}{1-\lambda_w}} X_t \right) \right] \\
& + (\xi_w \beta)^1 \, h_{j,t+1} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+1}^y\right)}{\left(1+\tau_{t+1}^w\right)} \frac{\psi_{z,t+1}^{\tau}}{\lambda_w} X_{t,1} + \zeta_{t+1} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+1}} X_{t,1} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+1} \right) \right] \\
& + (\xi_w \beta)^2 \, h_{j,t+2} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+2}^y\right)}{\left(1+\tau_{t+2}^w\right)} \frac{\psi_{z,t+2}^{\tau}}{\lambda_w} X_{t,2} + \zeta_{t+2} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+2}} X_{t,2} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+2} \right) \right] \\
& + (\xi_w \beta)^3 \, h_{j,t+3} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+3}^y\right)}{\left(1+\tau_{t+3}^w\right)} \frac{\psi_{z,t+3}^{\tau}}{\lambda_w} X_{t,3} + \zeta_{t+3} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+3}} X_{t,3} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+3} \right) \right] \\
+\ldots.. &= 0
\end{aligned}
$$

Consider first the case, $\xi_w = 0$. In this case, the above expression reduces to:

$$
0 = h_{j,t} \left[\tilde{w}_t w_t \frac{\psi_{z,t}}{\lambda_w} X_{t,0} + \zeta_t f_L \left((\tilde{w}_t X_{t,0})^{\frac{\lambda_w}{1-\lambda_w}} X_t \right) \right]
$$

or

$$
0 = h_{j,t} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+l}^y\right)}{\left(1+\tau_{t+l}^w\right)} \frac{\psi_{z,t}^{\tau}}{\lambda_w} X_{t,0} + \zeta_t f_L \left((\tilde{w}_t X_{t,0})^{\frac{\lambda_w}{1-\lambda_w}} X_t \right) \right].
$$

8.2 Linearizing the Wage Equation

8.2.1 Differentiate with respect to π_{t+l}, $l > 0$ and π_{t+l}, $l = 0$ evaluate that in steady state $\left(\tilde{w} = 1, \pi_t = \bar{\pi}, \quad w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w} + f_L = 0\right)$:

The first order condition that we want to linearize is given by:

$$
\begin{aligned}
F &= 0 = E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l h_{j,t+l} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+l}^y\right)}{\left(1+\tau_{t+l}^w\right)} \frac{\psi_{z,t+l}^\tau}{\lambda_w} \frac{\pi_t}{\pi_{t+l}} + \zeta_{t+l} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+l}} \frac{\pi_t}{\pi_{t+l}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+l} \right) \right] = \\
& h_{j,t} \left[\tilde{w}_t w_t \frac{\left(1-\tau_t^y\right)}{\left(1+\tau_t^w\right)} \frac{\psi_{z,t}^\tau}{\lambda_w} \frac{\pi_t}{\pi_t} + \zeta_t f_L \left(\left(\tilde{w}_t X_{t,0} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_t \right) \right] \\
& + (\xi_w \beta)^1 h_{j,t+1} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+1}^y\right)}{\left(1+\tau_{t+1}^w\right)} \frac{\psi_{z,t+1}^\tau}{\lambda_w} \frac{\pi_t}{\pi_{t+1}} + \zeta_{t+1} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+1}} \frac{\pi_t}{\pi_{t+1}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+1} \right) \right] \\
& + (\xi_w \beta)^2 h_{j,t+2} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+2}^y\right)}{\left(1+\tau_{t+2}^w\right)} \frac{\psi_{z,t+2}^\tau}{\lambda_w} \frac{\pi_t}{\pi_{t+2}} + \zeta_{t+2} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+2}} \frac{\pi_t}{\pi_{t+2}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+2} \right) \right] \\
& + (\xi_w \beta)^3 h_{j,t+3} \left[\tilde{w}_t w_t \frac{\left(1-\tau_{t+3}^y\right)}{\left(1+\tau_{t+3}^w\right)} \frac{\psi_{z,t+3}^\tau}{\lambda_w} \frac{\pi_t}{\pi_{t+3}} + \zeta_{t+3} f_L \left(\left(\frac{\tilde{w}_t w_t}{w_{t+3}} \frac{\pi_t}{\pi_{t+3}} \right)^{\frac{\lambda_w}{1-\lambda_w}} X_{t+3} \right) \right] \\
+\ldots\ldots &= 0.
\end{aligned}
$$

The derivative of F with respect to π_{t+l} when $l > 0$, evaluated in steady state, is:

$$
\begin{aligned}
h &= X = H \\
RB &: -(\xi_w \beta)^l X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right] \frac{1}{\bar{\pi}} \\
CEE/ACEL &: -(\beta \xi_w)^j L \left[w \frac{u_c}{\lambda_w} + f_{LL} \frac{\lambda_w}{1-\lambda_w} L \right] \frac{1}{\pi}
\end{aligned}
$$

The derivative of F with respect to π_t, evaluated in steady state, is:

$$\mathcal{F}_{\pi_t} = \frac{\beta\xi_w}{1-\beta\xi_w} X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right] \frac{1}{\bar{\pi}}$$

8.2.2 Differentiate with respect to $\tilde{w}_t$, and evaluate the result in steady state:

The derivative of $\mathcal{F}$ with respect to $\tilde{w}_t$, evaluated in steady state, is:

$$\mathcal{F}_{\tilde{w}_t} = 0 = E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right] \tag{257}$$

Use the fact that:

$$\begin{aligned}
\left(\sum_{j=0}^{\infty} (\beta\xi_w)^j \right) &= (\beta\xi_w)^0 + (\beta\xi_w)^1 + (\beta\xi_w)^2 + ... \\
&= 1 + (\beta\xi_w)^1 + (\beta\xi_w)^2 + ... \\
(\beta\xi_w) \left(\sum_{j=0}^{\infty} (\beta\xi_w)^j \right) &= (\beta\xi_w)^1 + (\beta\xi_w)^2 + ... \\
\left(\sum_{j=0}^{\infty} (\beta\xi_w)^j \right) - (\beta\xi_w) \left(\sum_{j=0}^{\infty} (\beta\xi_w)^j \right) &= 1 \\
(1 - (\beta\xi_w)) \left(\sum_{j=0}^{\infty} (\beta\xi_w)^j \right) &= 1 \\
\left(\sum_{j=0}^{\infty} (\beta\xi_w)^j \right) &= \frac{1}{(1 - (\beta\xi_w))}
\end{aligned}$$

to rewrite (257):

$$
\begin{aligned}
\digamma_{\tilde{w}_t} &= 0 = E_t \sum_{l=0}^{\infty} (\xi_w \beta)^l X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right] \\
&= \frac{1}{(1-(\beta\xi_w))} X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right]
\end{aligned}
$$

8.2.3 Differentiate with respect to w_t,and evaluate that in steady state:

The derivative of $\digamma$ with respect to w_t, evaluated in steady state, is:

$$
\begin{aligned}
&\Rightarrow \left\{ \tilde{w} = 1, \pi_t = \bar{\pi}, \quad w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + f_L = 0 \right\} \Rightarrow \\
\digamma_{w_t} &= X \left[\frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} \right] \\
&\quad + (\xi_w \beta)^1 X \left[\frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} \frac{1}{w_t} X \right] \\
&\quad + (\xi_w \beta)^2 X \left[\frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} \frac{1}{w_t} X \right] \\
&\quad + (\xi_w \beta)^3 X \left[\frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} \frac{1}{w_t} X \right] \\
+..... &= 0
\end{aligned}
$$

Use the following to rewrite $\digamma_{w_t}$.

$$
\begin{aligned}
\left(\sum_{j=1}^{\infty}(\beta\xi_w)^j\right) &= (\beta\xi_w)^1 + (\beta\xi_w)^2 + ... \\
&= (\beta\xi_w)^1 + (\beta\xi_w)^2 + ... \\
(\beta\xi_w)\left(\sum_{j=1}^{\infty}(\beta\xi_w)^j\right) &= (\beta\xi_w)^2 + (\beta\xi_w)^3 + ... \\
\left(\sum_{j=1}^{\infty}(\beta\xi_w)^j\right) - (\beta\xi_w)\left(\sum_{j=1}^{\infty}(\beta\xi_w)^j\right) &= (\beta\xi_w) \\
(1-(\beta\xi_w))\left(\sum_{j=1}^{\infty}(\beta\xi_w)^j\right) &= (\beta\xi_w) \\
\left(\sum_{j=1}^{\infty}(\beta\xi_w)^j\right) &= \frac{(\beta\xi_w)}{(1-(\beta\xi_w))}
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow \quad \left\{\tilde{w}=1, \pi_t=\bar{\pi}, \quad w\frac{u_c}{\lambda_w}+f_L=0\right\} \Rightarrow \\
F_{w_t} &= X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}\right] \\
&\quad + (\xi_w\beta)^1 X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right] \\
&\quad + (\xi_w\beta)^2 X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right] \\
&\quad + (\xi_w\beta)^3 X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right] \\
&\quad + ... \\
&= X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}\right] + \frac{(\beta\xi_w)}{(1-(\beta\xi_w))}X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right] \qquad (*)
\end{aligned}
$$

Use the following "accounting" to rewrite $(*)$

$$\begin{aligned}
\frac{1}{(1-(\beta\xi_w))}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w} &= \\
&= \left[\begin{array}{c} X\left[\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X\right] \\ +\beta\xi_w X\left[\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X\right] \\ +(\beta\xi_w)^3 X\left[\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X\right]+... \end{array}\right] \\
\frac{(\beta\xi_w)}{(1-(\beta\xi_w))}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w} &= \left[\begin{array}{c} \beta\xi_w X\left[\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X\right] \\ +(\beta\xi_w)^3 X\left[\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X\right]+... \end{array}\right] \\
\frac{1}{(1-(\beta\xi_w))}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w} &= \left[\begin{array}{c} \frac{(\beta\xi_w)}{(1-(\beta\xi_w))}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w} \\ -X\left[\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X\right] \end{array}\right] \\
&= \left[\begin{array}{c} \frac{(\beta\xi_w)}{(1-(\beta\xi_w))}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w} \\ -X\frac{u_c}{\lambda_w}-f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2 \end{array}\right] \\
&\quad \frac{1}{(1-(\beta\xi_w))}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w}+X\frac{u_c}{\lambda_w} \\
&= \frac{(\beta\xi_w)}{(1-(\beta\xi_w))}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w}-f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2
\end{aligned}$$

$$\begin{aligned}
& L\frac{u_c}{\lambda_w}+\frac{\beta\xi_w}{1-\beta\xi_w}X\left[\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X\right] \\
=\quad & \frac{1}{1-\beta\xi_w}X\left[w\frac{u_c}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{w}-f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2.
\end{aligned}$$

The result is:

$$\begin{aligned}
F_{w_t} &= X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}\right]+\frac{(\beta\xi_w)}{(1-(\beta\xi_w))}h\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right] \\
&= \frac{1}{1-\beta\xi_w}X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right]-\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2
\end{aligned}$$

8.2.4 Differentiate with respect to w_{t+l}, $l > 0$, and evaluate in steady state:

The derivative of $\mathcal{F}$ with respect to w_{t+l}, evaluated in steady state, is:

$$\begin{aligned}\mathcal{F}_{w_{t+l}} &= -\left(\xi_w \beta\right)^l h \left[\zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} \frac{1}{w} X\right] \\ &= -\left(\xi_w \beta\right)^l \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} \frac{1}{w} X^2\end{aligned}$$

8.2.5 Differentiate with respect to X_{t+l} (aggregate employment) and $\psi^{\tau}_{z,t+l}$, $l > 0$, and evaluate in steady state:

The derivative of $\mathcal{F}$ with respect to X_{t+l}, evaluated in steady state, is:

$$\begin{aligned}\mathcal{F}_{X_{t+l}} &= \left(\xi_w \beta\right)^l h_j \zeta f_{LL} \\ &= \left(\xi_w \beta\right)^l X \zeta f_{LL}\end{aligned}$$

The derivative of $\mathcal{F}$ with respect to $\psi^{\tau}_{z,t+l}$, evaluated in steady state, is:

$$\begin{aligned}\mathcal{F}_{\psi^{\tau}_{z,t+l}} &= \left(\beta \xi_w\right)^j h_j w \frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)} \frac{1}{\lambda_w} \\ &= \left(\beta \xi_w\right)^j X w \frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)} \frac{1}{\lambda_w}\end{aligned}$$

8.2.6 Differentiate with respect to ζ_{t+l} and evaluate in steady state:

The derivative of $\digamma$ with respect to ζ_{t+l}, evaluated in steady state, is:

$$\begin{aligned} \digamma_{\zeta_{t+l}} &= (\xi_w \beta)^l h_{j,t+l} f_L \\ &= (\xi_w \beta)^l X f_L \\ &= -X (\beta \xi_w)^l W \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} \end{aligned}$$

8.2.7 Differentiate with respect to τ_{t+l}^y and τ_{t+l}^w and evaluate in steady state:

The derivative of $\digamma$ with respect to τ_{t+l}^y and τ_{t+l}^w, evaluated in steady state, is:

$$\begin{aligned} \digamma_{\tau_{t+l}^y} &= (\xi_w \beta)^l X w \frac{1}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} \\ \digamma_{\tau_{t+l}^w} &= -(\xi_w \beta)^l X w \frac{(1-\tau^y)}{(1+\tau^w)^2} \frac{\psi_z^\tau}{\lambda_w} \end{aligned}$$

8.2.8 Collecting terms

$$
\begin{aligned}
\mathcal{F}_{\pi_{t+l}} &: \quad -\left(\xi_w\beta\right)^l h_j\left[w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{\bar{\pi}} \\
\mathcal{F}_{\pi_t} &: \quad \frac{\beta\xi_w}{1-\beta\xi_w}h\left[w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{\bar{\pi}} \\
\mathcal{F}_{\tilde{w}_t} &: \quad \frac{1}{(1-(\beta\xi_w))}X\left[w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right] \\
\mathcal{F}_{w_t} &: \quad \frac{1}{1-\beta\xi_w}X\left[\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right]-\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2 \\
\mathcal{F}_{w_{t+l}} &: \quad -\left(\xi_w\beta\right)^l\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2 \\
\mathcal{F}_{X_{t+l}} &: \quad \left(\xi_w\beta\right)^l X\zeta f_{LL} \\
\mathcal{F}_{\psi_{z,t+l}^\tau} &: \quad \left(\beta\xi_w\right)^j Xw\frac{(1-\tau^y)}{(1+\tau^w)}\frac{1}{\lambda_w} \\
\mathcal{F}_{\zeta_{t+l}} &: \quad \left(\xi_w\beta\right)^l Xf_L=-X\left(\beta\xi_w\right)^l W\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w} \\
\mathcal{F}_{\tau_{t+l}^y} &: \quad \left(\xi_w\beta\right)^l Xw\frac{1}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w} \\
\mathcal{F}_{\tau_{t+l}^w} &: \quad -\left(\xi_w\beta\right)^l Xw\frac{(1-\tau^y)}{(1+\tau^w)^2}\frac{\psi_z^\tau}{\lambda_w}
\end{aligned}
$$

Plug these into the formula (using the notation $\hat{x}_t=\frac{dX_t}{X}$):

$$\frac{a\mathcal{F}_a}{\mathcal{F}}\hat{a}+\frac{b\mathcal{F}_b}{\mathcal{F}}\hat{b}+...=0$$

since $\frac{a\mathcal{F}_a}{\mathcal{F}}\hat{a}+\frac{b\mathcal{F}_b}{\mathcal{F}}\hat{b}+...=0$ we have:

$$a\mathcal{F}_a\hat{a}+b\mathcal{F}_b\hat{b}+...=0$$

Use this expression to get:

$$
\begin{aligned}
&-\left(\xi_w\beta\right)^l X\left[w\frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{\bar{\pi}}\pi_{t+l}\hat{\pi}_{t+l} \\
&+\frac{\beta\xi_w}{1-\beta\xi_w}X\left[w\frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\frac{1}{\bar{\pi}}\pi_t\hat{\pi}_t \\
&+\frac{1}{\left(1-\left(\beta\xi_w\right)\right)}X\left[w\frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\tilde{w}_t\hat{\tilde{w}}_t \\
&+\frac{1}{1-\beta\xi_w}X\left[\frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w_t}X\right]w_t\hat{w}_t-\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2w_t\hat{w}_t \\
&-\left(\xi_w\beta\right)^l\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}\frac{1}{w}X^2w_{t+l}\hat{w}_{t+l} \\
&+\left(\xi_w\beta\right)^l X\zeta f_{LL}X_{t+l}\hat{X}_{t+l} \\
&+\left(\beta\xi_w\right)^j Xw\frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)}\frac{1}{\lambda_w}\psi_{z,t+l}^\tau\hat{\psi}_{z,t+l}^\tau \\
&-X\left(\beta\xi_w\right)^l W\frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)}\frac{\psi_z^\tau}{\lambda_w}\zeta_{t+l}\hat{\zeta}_{t+l} \\
&+\left(\xi_w\beta\right)^l Xw\frac{1}{\left(1+\tau^w\right)}\frac{\psi_z^\tau}{\lambda_w}\tau_{t+l}^y\hat{\tau}_{t+l}^y \\
&-\left(\xi_w\beta\right)^l Xw\frac{\left(1-\tau^y\right)}{\left(1+\tau^w\right)^2}\frac{\psi_z^\tau}{\lambda_w}\tau_{t+l}^w\hat{\tau}_{t+l}^w
\end{aligned}
$$

or

$$
\begin{aligned}
0 \quad = \quad & \frac{1}{(1-\beta\xi_w)} X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right] \left(\widehat{\tilde{w}}_t + \hat{w}_t \right) \\
& - \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X^2 \sum_{j=0}^{\infty} (\xi_w \beta)^l \hat{w}_{t+l} \\
& - \sum_{j=1}^{\infty} (\xi_w \beta)^l X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right] \hat{\pi}_{t+l} \\
& + \frac{\beta\xi_w}{1-\beta\xi_w} X \left[w \frac{(1-\tau^y)}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} + \zeta f_{LL} \frac{\lambda_w}{1-\lambda_w} X \right] \hat{\pi}_t \\
& + \zeta f_{LL} X^2 \sum_{j=0}^{\infty} (\xi_w \beta)^l \hat{X}_{t+l} + \frac{(1-\tau^y)}{(1+\tau^w)} \frac{X w \psi_z^\tau}{\lambda_w} \sum_{j=0}^{\infty} (\beta \xi_w)^j \hat{\psi}_{z,t+l}^{\tau} \\
& - \frac{(1-\tau^y)}{(1+\tau^w)} \frac{X w \psi_z^\tau}{\lambda_w} \zeta \sum_{j=0}^{\infty} (\beta \xi_w)^l \hat{\zeta}_{t+l} \\
& + X w \frac{1}{(1+\tau^w)} \frac{\psi_z^\tau}{\lambda_w} \tau^y \sum_{j=0}^{\infty} (\xi_w \beta)^l \hat{\tau}_{t+l}^y \\
& - X w \frac{(1-\tau^y)}{(1+\tau^w)^2} \frac{\psi_z^\tau}{\lambda_w} \tau^w \sum_{j=0}^{\infty} (\xi_w \beta)^l \hat{\tau}_{t+l}^w
\end{aligned}
$$

After dividing by X, this expression is written:

$$
\begin{aligned}
0 \quad = \quad & \frac{1}{(1-\beta\xi_w)}\left[w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\left(\widehat{\tilde{w}}_t+\hat{w}_t\right)-\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{w}_{t+l} \\
& -\sum_{j=1}^{\infty}(\xi_w\beta)^l\left[w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\hat{\pi}_{t+l} \\
& +\frac{\beta\xi_w}{1-\beta\xi_w}\left[w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\hat{\pi}_t \\
& +\zeta f_{LL}X\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{X}_{t+l}+\frac{(1-\tau^y)}{(1+\tau^w)}\frac{w\psi_z^\tau}{\lambda_w}\sum_{j=0}^{\infty}(\beta\xi_w)^j\,\hat{\psi}_{z,t+l}^{\tau} \\
& -\frac{(1-\tau^y)}{(1+\tau^w)}\frac{w\psi_z^\tau}{\lambda_w}\zeta\sum_{j=0}^{\infty}(\beta\xi_w)^l\,\hat{\zeta}_{t+l} \\
& +\frac{\tau^y}{(1+\tau^w)}\frac{w\psi_z^\tau}{\lambda_w}\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{\tau}_{t+l}^{y} \\
& -\frac{\tau^w(1-\tau^y)}{(1+\tau^w)(1+\tau^w)}\frac{w\psi_z^\tau}{\lambda_w}\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{\tau}_{t+l}^{w}
\end{aligned}
$$

Denoting $\left[w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]=\left[-f_L+f_{LL}\frac{\lambda_w}{1-\lambda_w}X\right]\equiv\tilde{\sigma}_L$,$w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^\tau}{\lambda_w}=-f_L$ we obtain:

$$
\begin{aligned}
0 \quad = \quad & \frac{1}{(1-\beta\xi_w)}\tilde{\sigma}_L\left(\widehat{\tilde{w}}_t+\hat{w}_t\right)-\zeta f_{LL}\frac{\lambda_w}{1-\lambda_w}X\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{w}_{t+l} \\
& -\sum_{j=1}^{\infty}(\xi_w\beta)^l\,\tilde{\sigma}_L\hat{\pi}_{t+l}+\frac{\beta\xi_w}{1-\beta\xi_w}\tilde{\sigma}_L\hat{\pi}_t \\
& +\zeta f_{LL}X\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{X}_{t+l}-f_L\sum_{j=0}^{\infty}(\beta\xi_w)^j\,\hat{\psi}_{z,t+l}^{\tau} \\
& +f_L\sum_{j=0}^{\infty}(\beta\xi_w)^l\,\hat{\zeta}_{t+l} \\
& -\frac{\tau^y}{(1-\tau^y)}f_L\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{\tau}_{t+l}^{y}+\frac{\tau^w}{(1+\tau^w)}f_L\sum_{j=0}^{\infty}(\xi_w\beta)^l\,\hat{\tau}_{t+l}^{w}
\end{aligned}
$$

This can be written: (with $\zeta=1$)

$$
\begin{aligned}
&\frac{1}{(1-\beta\xi_w)}\tilde{\sigma}_L\left(\widehat{\tilde{w}}_t+\hat{w}_t\right)\\
=\;&\left[\begin{array}{c}
\tilde{\sigma}_L\sum_{j=1}^{\infty}\left(\xi_w\beta\right)^l\hat{\pi}_{t+l}-f_L\sum_{j=0}^{\infty}\left(\beta\xi_w\right)^j\hat{\psi}^{\tau}_{z,t+l}+f_{LL}X\frac{\lambda_w}{1-\lambda_w}\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{w}_{t+l}\\
-f_{LL}X\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{X}_{t+l}-\frac{\beta\xi_w}{1-\beta\xi_w}\tilde{\sigma}_L\hat{\pi}_t+w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^{\tau}}{\lambda_w}\zeta\sum_{j=0}^{\infty}\left(\beta\xi_w\right)^l\hat{\zeta}_{t+l}\\
-\frac{\tau^y}{(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{\tau}^y_{t+l}+\frac{\tau^w(1-\tau^y)}{(1+\tau^w)(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{\tau}^w_{t+l}
\end{array}\right]\\
=\;&\left[\begin{array}{c}
\tilde{\sigma}_L\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{\pi}_{t+l}-f_L\sum_{j=0}^{\infty}\left(\beta\xi_w\right)^j\hat{\psi}^{\tau}_{z,t+l}+f_{LL}X\frac{\lambda_w}{1-\lambda_w}\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{w}_{t+l}\\
-f_{LL}X\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{X}_{t+l}+w\frac{(1-\tau^y)}{(1+\tau^w)}\frac{\psi_z^{\tau}}{\lambda_w}\zeta\sum_{j=0}^{\infty}\left(\beta\xi_w\right)^l\hat{\zeta}_{t+l}\\
-\frac{\tau^y}{(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{\tau}^y_{t+l}+\frac{\tau^w(1-\tau^y)}{(1+\tau^w)(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\sum_{j=0}^{\infty}\left(\xi_w\beta\right)^l\hat{\tau}^w_{t+l}\\
-\left[\tilde{\sigma}_L+\frac{\beta\xi_w}{1-\beta\xi_w}\tilde{\sigma}_L\right]\hat{\pi}_t
\end{array}\right]\\
=\;&\sum_{l=0}^{\infty}(\beta\xi_w)^l\left[\begin{array}{c}
\tilde{\sigma}_L\hat{\pi}_{t+l}+f_L\hat{\psi}^{\tau}_{z,t+l}+f_{LL}L\frac{\lambda_w}{1-\lambda_w}\hat{w}_{t+l}-f_{LL}X\hat{X}_{t+j}+\frac{(1-\tau^y)}{(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\hat{\zeta}_{t+j}\\
-\frac{\tau^y}{(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\hat{\tau}^y_{t+l}+\frac{\tau^w(1-\tau^y)}{(1+\tau^w)(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\hat{\tau}^w_{t+l}
\end{array}\right]-\frac{\tilde{\sigma}_L}{1-\beta\xi_w}\hat{\pi}_t \qquad (\S)
\end{aligned}
$$

$$
\left[\tilde{\sigma}_L+\frac{\beta\xi_w}{1-\beta\xi_w}\tilde{\sigma}_L\right]=\frac{\tilde{\sigma}_L}{1-\beta\xi_w}
$$

Consider the following example:

$$
x_t+z_t=\sum_{l=0}^{\infty}(\beta\xi_w)^l y_{t+l}.
$$

Also,

$$
\beta\xi_w\left(x_{t+1}+z_{t+1}\right)=\sum_{l=1}^{\infty}(\beta\xi_w)^l y_{t+l}
$$

Differencing these expressions:

$$
x_t+z_t-\beta\xi_w\left(x_{t+1}+z_{t+1}\right)=y_t.
$$

Applying this to (§), we get:

$$\frac{1}{(1-\beta\xi_w)}\tilde{\sigma}_L\left(\widehat{\tilde{w}}_t+\hat{w}_t\right)$$

$$=\left[\begin{array}{c}\beta\xi_w\frac{1}{(1-\beta\xi_w)}\tilde{\sigma}_L\left(\widehat{\tilde{w}}_{t+1}+\hat{w}_{t+1}\right)+\tilde{\sigma}_L\beta\xi_w\hat{\pi}_{t+1}\\ +\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\left(\beta\xi_w\hat{\pi}_{t+1}-\hat{\pi}_t\right)\\ +f_L\hat{\psi}^{\tau}_{z,t}+f_{LL}L\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}X\hat{X}_t+\frac{(1-\tau^y)}{(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\hat{\zeta}_t\\ -\frac{\tau^y}{(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\hat{\tau}_t^y+\frac{\tau^w(1-\tau^y)}{(1+\tau^w)(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w}\hat{\tau}_t^w\end{array}\right]$$

Note that

$$\begin{aligned}\frac{\tau^y}{(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w} &= \frac{\tau^y}{(1+\tau^w)}\frac{(1-\tau^y)}{(1-\tau^y)}\frac{w\psi_z^{\tau}}{\lambda_w}=-\frac{\tau^y}{(1-\tau^y)}f_L,\\ \frac{\tau^w(1-\tau^y)}{(1+\tau^w)(1+\tau^w)}\frac{w\psi_z^{\tau}}{\lambda_w} &= -\frac{\tau^w}{(1+\tau^w)}f_L.\end{aligned}$$

Use the above to get:

$$\frac{1}{(1-\beta\xi_w)}\tilde{\sigma}_L\left(\widehat{\tilde{w}}_t+\hat{w}_t\right)$$

$$=\left[\begin{array}{c}\beta\xi_w\frac{1}{(1-\beta\xi_w)}\tilde{\sigma}_L\left(\widehat{\tilde{w}}_{t+1}+\hat{w}_{t+1}\right)+\tilde{\sigma}_L\beta\xi_w\hat{\pi}_{t+1}\\ +\tilde{\sigma}_L\frac{\beta\xi_w}{1-\beta\xi_w}\left(\beta\xi_w\hat{\pi}_{t+1}-\hat{\pi}_t\right)\\ +f_L\hat{\psi}^{\tau}_{z,t}+f_{LL}X\frac{\lambda_w}{1-\lambda_w}\hat{w}_t-f_{LL}X\hat{X}_t-f_L\hat{\zeta}_t\\ +\frac{\tau^y}{(1-\tau^y)}f_L\hat{\tau}_t^y-\frac{\tau^w}{(1+\tau^w)}f_L\hat{\tau}_t^w\end{array}\right]$$

Note that

$$\sigma_L=\frac{f_{LL}X}{f_L},$$

since

$$\frac{f_{LL}(X)}{f_L(X)}=\frac{-\sigma_L A_L X^{\sigma_L-1}}{-A_L X^{\sigma_L}}=\frac{\sigma_L}{X}.$$

Note, that $\tilde{\sigma}_L=-f_L+f_{LL}\frac{\lambda_w}{1-\lambda_w}X=f_L\left[\frac{\lambda_w}{1-\lambda_w}\frac{f_{LL}X}{f_L}-1\right]=f_L\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L-1\right],$

so dividing the expression above by f_L

$$\frac{1}{(1-\beta\xi_w)}\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right]\left(\widehat{\tilde{w}}_t + \hat{w}_t\right) \tag{258}$$

$$= \left[\begin{array}{c} \frac{\beta\xi_w}{(1-\beta\xi_w)}\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right]\left(\widehat{\tilde{w}}_{t+1} + \hat{w}_{t+1}\right) + \left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right]\beta\xi_w\hat{\pi}_{t+1} \\ -\frac{\beta\xi_w}{1-\beta\xi_w}\left[\frac{\lambda_w}{1-\lambda_w}\sigma_L - 1\right](\hat{\pi}_t - \beta\xi_w\hat{\pi}_{t+1}) \\ +\hat{\psi}^{\tau}_{z,t} + \sigma_L\frac{\lambda_w}{1-\lambda_w}\hat{w}_t - \sigma_L\hat{X}_t \\ +\frac{\tau^y}{(1-\tau^y)}\hat{\tau}^y_t - \frac{\tau^w}{(1+\tau^w)}\hat{\tau}^w_t - \hat{\zeta}_t \end{array}\right]$$

8.2.9 The aggregate wage equation.

The aggregate wage equation is given by:

$$W_t = \left[(1-\xi_w)\left(\tilde{W}_t\right)^{\frac{1}{1-\lambda_w}} + \xi_w\left(\pi_{t-1}\mu_{z,t}W_{t-1}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}.$$

Dividing this by $z_t P_t$, we get:

$$w_t = \left[(1-\xi_w)(\tilde{w}_t w_t)^{\frac{1}{1-\lambda_w}} + \xi_w\left(\frac{\pi_{t-1}}{\pi_t}w_{t-1}\right)^{\frac{1}{1-\lambda_w}}\right]^{1-\lambda_w}.$$

(It is easy to see from this that the steady state value of $\tilde{w}_t$ is unity.) w_t is the scaled real wage. linearizing this leads to:

$$\hat{w}_t = (1-\xi_w)\left(\hat{w}_t + \widehat{\tilde{w}}_t\right) + \xi_w\left(\hat{w}_{t-1} - (\hat{\pi}_t - \hat{\pi}_{t-1})\right)$$

or

$$(1-\xi_w)\left(\hat{w}_t + \widehat{\tilde{w}}_t\right) = \hat{w}_t - \xi_w\left(\hat{w}_{t-1} - (\hat{\pi}_t - \hat{\pi}_{t-1})\right)$$

Subsituting this into (258), and multiply by $1-\lambda_w$:

$$
\begin{aligned}
& \frac{1}{(1-\beta\xi_w)(1-\xi_w)}\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\left[\hat{w}_t-\xi_w\left(\hat{w}_{t-1}-(\hat{\pi}_t-\hat{\pi}_{t-1})\right)\right] \qquad (259) \\
= & \left[\begin{array}{c}
\frac{\beta\xi_w}{(1-\beta\xi_w)(1-\xi_w)}\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\left[\hat{w}_{t+1}-\xi_w\left(\hat{w}_t-(\hat{\pi}_{t+1}-\hat{\pi}_t)\right)\right] \\
+\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\beta\xi_w\hat{\pi}_{t+1}+(1-\lambda_w)\hat{\psi}^{\tau}_{z,t} \\
+\sigma_L\lambda_w\hat{w}_t-(1-\lambda_w)\sigma_L\hat{X}_t \\
-\frac{\beta\xi_w}{1-\beta\xi_w}\left[\lambda_w\sigma_L-(1-\lambda_w)\right]\left(\hat{\pi}_t-\beta\xi_w\hat{\pi}_{t+1}\right) \\
+(1-\lambda_w)\frac{\tau^y}{(1-\tau^y)}\hat{\tau}^y_t-(1-\lambda_w)\frac{\tau^w}{(1+\tau^w)}\hat{\tau}^w_t-(1-\lambda_w)\hat{\zeta}_t
\end{array}\right]
\end{aligned}
$$

Letting $b_w=\frac{[\lambda_w\sigma_L-(1-\lambda_w)]}{[(1-\beta\xi_w)(1-\xi_w)]}$, we obtain

$$
\begin{aligned}
& \hat{w}_tb_w-\xi_w\hat{w}_{t-1}b_w+\xi_wb_w\hat{\pi}_t-\xi_wb_w\hat{\pi}_{t-1} \qquad (260) \\
= & \left[\begin{array}{c}
\beta\xi_wb_w\hat{w}_{t+1}-\beta\xi_wb_w\xi_w\hat{w}_t+\beta\xi_wb_w\xi_w\hat{\pi}_{t+1}-\beta\xi_wb_w\xi_w\hat{\pi}_t \\
+b_w(1-\xi_w)(1-\beta\xi_w)\beta\xi_w\hat{\pi}_{t+1}+(1-\lambda_w)\hat{\psi}^{\tau}_{z,t} \\
+\sigma_L\lambda_w\hat{w}_t-(1-\lambda_w)\sigma_L\hat{X}_t \\
-\beta\xi_wb_w(1-\xi_w)\hat{\pi}_t+\beta\xi_wb_w(1-\xi_w)\beta\xi_w\hat{\pi}_{t+1} \\
+(1-\lambda_w)\frac{\tau^y}{(1-\tau^y)}\hat{\tau}^y_t-(1-\lambda_w)\frac{\tau^w}{(1+\tau^w)}\hat{\tau}^w_t-(1-\lambda_w)\hat{\zeta}_t
\end{array}\right]
\end{aligned}
$$

or

$$
\begin{aligned}
0 = \; & \hat{w}_tb_w-\xi_w\hat{w}_{t-1}b_w+\xi_wb_w\hat{\pi}_t-\xi_wb_w\hat{\pi}_{t-1} \qquad (261) \\
& -\beta\xi_wb_w\hat{w}_{t+1}+\beta\xi_wb_w\xi_w\hat{w}_t-\beta\xi_wb_w\xi_w\hat{\pi}_{t+1}+\beta\xi_wb_w\xi_w\hat{\pi}_t \\
& -b_w(1-\xi_w)(1-\beta\xi_w)\beta\xi_w\hat{\pi}_{t+1}-(1-\lambda_w)\hat{\psi}^{\tau}_{z,t} \\
& -\sigma_L\lambda_w\hat{w}_t+(1-\lambda_w)\sigma_L\hat{X}_t \\
& +\beta\xi_wb_w(1-\xi_w)\hat{\pi}_t-\beta\xi_wb_w(1-\xi_w)\beta\xi_w\hat{\pi}_{t+1} \\
& -(1-\lambda_w)\frac{\tau^y}{(1-\tau^y)}\hat{\tau}^y_t+(1-\lambda_w)\frac{\tau^w}{(1+\tau^w)}\hat{\tau}^w_t+(1-\lambda_w)\hat{\zeta}_t
\end{aligned}
$$

or

$$
\begin{aligned}
0 \quad = \quad & b_w \xi_w \hat{w}_{t-1} \\
& - \left(\left(1 + \beta \xi_w^2 \right) b_w + \lambda_w \sigma_L \right) \hat{w}_t \\
& + \beta \xi_w b_w \hat{w}_{t+1} \\
& + \xi_w b_w \hat{\pi}_{t-1} \\
& - (1 + \beta) \xi_w b_w \hat{\pi}_t \\
& + b_w \beta \xi_w \pi_{t+1} \\
& + (1 - \lambda_w) \hat{\psi}_{z,t}^{\tau} \\
& - (1 - \lambda_w) \sigma_L \hat{X}_t \\
& - (1 - \lambda_w) \hat{\zeta}_t \\
& + (1 - \lambda_w) \frac{\tau^y}{(1 - \tau^y)} \hat{\tau}_t^y \\
& - (1 - \lambda_w) \frac{\tau^w}{(1 + \tau^w)} \hat{\tau}_t^w
\end{aligned}
$$

Reintroduce the expectations operator:

$$
E_t \left\{ \eta_0 \hat{w}_{t-1} + \eta_1 \hat{w}_t + \eta_2 \hat{w}_{t+1} + \eta_3^- \hat{\pi}_{t-1} + \eta_3 \hat{\pi}_t + \eta_4 \hat{\pi}_{t+1} + \eta_5 \hat{H}_t + \eta_6 \hat{\psi}_{z,t} + \eta_7 \hat{\zeta}_t + \eta_8 \hat{\tau}_t^y + \eta_9 \hat{\tau}_t^w \right\} = 0
$$

where

$$\eta = \begin{pmatrix} b_w \xi_w \\ -b_w \left(1 + \beta \xi_w^2\right) + \sigma_L \lambda_w \\ \beta \xi_w b_w \\ b_w \xi_w \\ -\xi_w b_w (1+\beta) \\ b_w \beta \xi_w \\ -(1-\lambda_w) \sigma_L \\ (1-\lambda_w) \\ -(1-\lambda_w) \\ (1-\lambda_w) \frac{\tau^y}{(1-\tau^y)} \\ -(1-\lambda_w) \frac{\tau^w}{(1+\tau^w)} \end{pmatrix} = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_{\bar{3}} \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \\ \eta_8 \\ \eta_9 \end{pmatrix}.$$

Note that we have used H instead of X in the equation above. This is due to the fact that in ss all households (intermediate firms) supply (demand) the same amount of labor. Compare with $ACEL$ where L equals our H:

(3) $E_t \left\{ \eta_0 \hat{w}_{t-1} + \eta_1 \hat{w}_t + \eta_2 \hat{w}_{t+1} + \eta_{\bar{3}} \hat{\pi}_{t-1} + \eta_3 \hat{\pi}_t + \eta_4 \hat{\pi}_{t+1} + \eta_5 \hat{L}_t + \eta_6 \hat{\psi}_{z,t} + \eta_7 \hat{\zeta}_t \right\} = 0$

where

$$\eta = \begin{pmatrix} b_w \xi_w \\ -b_w \left(1 + \beta \xi_w^2\right) + \sigma_L \lambda_w \\ \beta \xi_w b_w \\ b_w \xi_w \\ -\xi_w b_w (1+\beta) \\ b_w \beta \xi_w \\ -\sigma_L (1-\lambda_w) \\ 1-\lambda_w \\ -(1-\lambda_w) \end{pmatrix} = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_{\bar{3}} \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \\ \eta_7 \end{pmatrix}.$$

9 Net foreign assets

The evolution of net foreign assets at the aggregate level satisfies

$$S_t B^*_{t+1} = S_t P^*_t Y^*_t - S_t P^*_t \left(C^m_t + I^m_t \right) + R^*_{t-1} \Phi(a_{t-1}, \widetilde{\phi}_t) S_t B^*_t,$$

where we notice that $R^*_{t-1} \Phi(a_{t-1}, \widetilde{\phi}_t)$ is the risk-adjusted gross nominal interest rate. Note that the definition of a_t is given by

$$a_t \equiv \frac{S_t B^*_{t+1}}{P_t z_t}.$$

Dividing through with $1/(P_t z_t)$, we obtain

$$\frac{S_t B^*_{t+1}}{P_t z_t} = \frac{S_t P^*_t}{P_t} \frac{Y^*_t}{z_t} - \frac{S_t P^*_t}{P_t} \left(\frac{C^m_t}{z_t} + \frac{I^m_t}{z_t} \right) + R^*_{t-1} \Phi(a_{t-1}, \widetilde{\phi}_t) \frac{S_{t-1} B^*_t}{P_{t-1} z_{t-1}} \frac{S_t P_{t-1} z_{t-1}}{S_{t-1} P_t z_t},$$

and using our definition of a_t and letting lower case variables denote detrended values, we have

$$a_t = \gamma^x_t y^*_t - \gamma^x_t \left(c^m_t + i^m_t \right) + \left(1 + R^*_{t-1} \Phi(a_{t-1}, \widetilde{\phi}_t) \right) \frac{a_{t-1}}{\pi_t \mu_{z,t}} \frac{S_t}{S_{t-1}}.$$

Totally differentiating this expression and using that $a = 0$, $\Phi(0,0) = 1$, $R^* = R$ and $\frac{\bar{S}_t}{\bar{S}_{t-1}} = 1$ in steady state, we derive

$$da_t = d\gamma^x_t y^* + \gamma^x dy^*_t - \left(c^m + i^m \right) d\gamma^x_t - \gamma^x \left(dc^m_t + di^m_t \right) + \frac{R}{\pi \mu_z} da_{t-1},$$

and log-linearizing this equation, we obtain

$$\hat{a}_t = y^* \gamma^x \left(\hat{\gamma}^x_t + \hat{y}^*_t \right) - \left(c^m + i^m \right) \gamma^x \hat{\gamma}^x_t - \gamma^x \left(c^m \hat{c}^m_t + i^m \hat{i}^m_t \right) + \frac{R}{\pi \mu_z} \hat{a}_{t-1},$$

and using the expressions for $\hat{c}^m_t$ and $\hat{i}^m_t$,[3] we finally derive

$$\hat{a}_t = y^* \gamma^x \left(\hat{\gamma}^x_t + \hat{y}^*_t \right) - \left(c^m + i^m \right) \gamma^x \hat{\gamma}^x_t - \gamma^x \left(\begin{array}{c} c^m \left(\widehat{c}_t - \eta_c (1 - \omega_c) \left(\gamma^{c,d} \right)^{-(1-\eta_c)} \widehat{\gamma}^{mc,d}_t \right) + \\ i^m \left(\widehat{i}_t - \eta_i (1 - \omega_i) \left(\gamma^{i,d} \right)^{-(1-\eta_i)} \widehat{\gamma}^{mi,d}_t \right) \end{array} \right) + \frac{R}{\pi \mu_z} \hat{a}_{t-1}$$

[3] $\hat{c}^m_t =$

which is the log-linearized equation for net foreign assets in this model. If we instead use $\gamma^x = \frac{1}{\gamma^f}$ and $\hat{\gamma}_t^x = -\hat{\gamma}_t^f$ we get:

$$\hat{a}_t = y^* \frac{1}{\gamma^f}\left(-\hat{\gamma}_t^f + \hat{y}_t^*\right) + (c^m + i^m)\frac{1}{\gamma^f}\hat{\gamma}_t^f - \frac{1}{\gamma^f}\left(\begin{array}{c} c^m\left(\widehat{c}_t - \eta_c(1-\omega_c)\left(\gamma^{c,d}\right)^{-(1-\eta_c)}\widehat{\gamma}_t^{mc,d}\right) + \\ i^m\left(\widehat{i}_t - \eta_i(1-\omega_i)\left(\gamma^{i,d}\right)^{-(1-\eta_i)}\widehat{\gamma}_t^{mi,d}\right) \end{array}\right) + \frac{R}{\pi\mu_z}\hat{a}_{t-1}$$

9.1 Log-linearization of the Law of Motion for the Capital Stock

The law of motion for capital is

$$\begin{aligned} ACEL &: \bar{K}_{t+1} = (1-\delta)\bar{K}_t + \mu_{\Upsilon,t}\Upsilon^t F(I_t, I_{t-1}) + \Delta_t \\ RB &: \bar{K}_{t+1} = (1-\delta)\bar{K}_t + \Upsilon_t F(I_t, I_{t-1}) + \Delta_t \end{aligned}$$

Scale the capital stock by z_t :

$$\bar{k}_{t+1} = \frac{\bar{K}_{t+1}}{z_t}.$$

Also, recall that investment is scaled as follows:

$$i_t = \frac{I_t}{z_t}$$

we obtain

$$\frac{\bar{K}_{t+1}}{z_t} z_t = (1-\delta)\frac{\bar{K}_t}{z_{t-1}} z_{t-1} + \Upsilon_t \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t z_t + \Delta_t$$

or

$$\bar{k}_{t+1} z_t = (1-\delta)\bar{k}_t z_{t-1} + \Upsilon_t \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t z_t + \Delta_t$$

or

$$\bar{k}_{t+1} = (1-\delta)\bar{k}_t \frac{z_{t-1}}{z_t} + \Upsilon_t \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t + \frac{\Delta_t}{z_t}$$

or (denoting $\bar{\Delta}_t = \frac{\Delta_t}{z_t}$

$$\bar{k}_{t+1} = (1-\delta)\bar{k}_t \frac{1}{\mu_{z,t}} + \Upsilon_t \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t + \bar{\Delta}_t \qquad (\#)$$

Note that, given our assumption that $S = 0$ in steady state,

$$\begin{aligned} \bar{k} &= (1-\delta)\bar{k}\frac{1}{\mu_z} + i \\ 1 &= (1-\delta)\frac{1}{\mu_z} + \frac{i}{\bar{k}} \\ \frac{i}{\bar{k}} &= \left(1 - (1-\delta)\frac{1}{\mu_z}\right) \end{aligned}$$

Linearizing the preceding expression (#), and evaluating the result in steady state (taking into account $S = S' = 0$) :

Differentiate the left hand side:

$$\frac{\partial \bar{k}_{t+1}}{\partial \bar{k}_{t+1}} = 1$$

Differentiate the right hand side:

$$\begin{aligned} & F\left(\bar{k}_t, \mu_{z,t}, \Upsilon_t, i_t, i_{t-1}, \bar{\Delta}_t\right) \\ = \; & (1-\delta)\bar{k}_t \frac{1}{\mu_{z,t}} + \Upsilon_t \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t + \bar{\Delta}_t \end{aligned}$$

$$\begin{aligned} & F_1\left(\bar{k}_t, \mu_{z,t}, \Upsilon_t, i_t, i_{t-1}\right) \\ = \; & (1-\delta)\frac{1}{\mu_{z,t}} \\ ss \; : \; & (1-\delta)\frac{1}{\mu_z} \end{aligned}$$

$$
\begin{aligned}
& F_2\left(\bar{k}_t, \mu_{z,t}, \Upsilon_t, i_t, i_{t-1}\right) \\
= \quad & -(1-\delta)\bar{k}_t \frac{1}{\mu_{z,t}^2} - \Upsilon_t S'(\frac{i_t \mu_{z,t}}{i_{t-1}}) i_t \frac{i_t}{i_{t-1}} \\
ss \quad : \quad & -(1-\delta)\bar{k} \frac{1}{\mu_z^2}
\end{aligned}
$$

$$
\begin{aligned}
& F_3\left(\bar{k}_t, \mu_{z,t}, \Upsilon_t, i_t, i_{t-1}\right) \\
= \quad & \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] i_t \\
ss \quad : \quad & i
\end{aligned}
$$

$$
\begin{aligned}
& F_4\left(\bar{k}_t, \mu_{z,t}, \Upsilon_t, i_t, i_{t-1}\right) \\
= \quad & \Upsilon_t \left[1 - S(\frac{i_t \mu_{z,t}}{i_{t-1}})\right] - S'\left(\frac{i_t \mu_{z,t}}{i_{t-1}}\right) \frac{\mu_{z,t}}{i_{t-1}} \\
ss \quad : \quad & 1
\end{aligned}
$$

$$
\begin{aligned}
& F_5\left(\bar{k}_t, \mu_{z,t}, \Upsilon_t, i_t, i_{t-1}\right) \\
= \quad & \Upsilon_t i_t S'\left(\frac{i_t \mu_{z,t}}{i_{t-1}}\right) \frac{i_t \mu_{z,t}}{i_{t-1}^2} \\
ss \quad : \quad & 0
\end{aligned}
$$

$$
\begin{aligned}
& F_6\left(\bar{k}_t, \mu_{z,t}, \Upsilon_t, i_t, i_{t-1}, \bar{\Delta}_t\right) \\
= \quad & 1
\end{aligned}
$$

Collect terms:

$$\bar{k}\widehat{\bar{k}}_{t+1} = (1-\delta)\frac{1}{\mu_z}\bar{k}\widehat{\bar{k}}_t - (1-\delta)\bar{k}\frac{1}{\mu_z}\hat{\mu}_{z,t} + i\hat{\Upsilon}_t + i\hat{\imath}_t$$

or (divide by $\bar{k}$)

$$\widehat{\bar{k}}_{t+1} = (1-\delta)\frac{1}{\mu_z}\widehat{\bar{k}}_t - (1-\delta)\frac{1}{\mu_z}\hat{\mu}_{z,t} + \frac{i}{\bar{k}}\hat{\Upsilon}_t + \frac{i}{\bar{k}}\hat{\imath}_t$$

Making use of the expression for the steady state investment capital ratio:

$$\widehat{\bar{k}}_{t+1} = (1-\delta)\frac{1}{\mu_z}\widehat{\bar{k}}_t - (1-\delta)\frac{1}{\mu_z}\hat{\mu}_{z,t} + \left(1-(1-\delta)\frac{1}{\mu_z}\right)\hat{\Upsilon}_t + \left(1-(1-\delta)\frac{1}{\mu_z}\right)\hat{\imath}_t$$

Compare with $ACEL$:

$$\widehat{\bar{k}}_{t+1} = (1-\delta)\frac{1}{\mu_{z,t}\Upsilon^{\frac{1}{1-\theta}}}\widehat{\bar{k}}_t - (1-\delta)\frac{1}{\mu_{z,t}\Upsilon^{\frac{1}{1-\theta}}}\hat{\mu}_{z,t} + \left[1-\frac{1-\delta}{\mu_z\Upsilon^{\frac{1}{1-\theta}}}\right]\hat{\imath}_t + \left[1-\frac{1-\delta}{\mu_z\Upsilon^{\frac{1}{1-\theta}}}\right]\hat{\mu}_{\Upsilon,t}$$

9.2 Log-linearization of the Production Function

$$Y_{it} = z_t^{1-\alpha}\epsilon_t K_{i,t}^{\alpha} X_{i,t}^{1-\alpha} - z_t\phi \tag{262}$$

Stationarize:

$$\frac{Y_{it}}{z_t} = \frac{z_t^{1-\alpha}\epsilon_t K_{i,t}^{\alpha} X_{i,t}^{1-\alpha} z_{t-1}^{\alpha}}{z_t z_{t-1}^{\alpha}} - \phi \tag{263}$$

or

$$\begin{aligned}
y_{it} &= \frac{z_t^{1-\alpha}\epsilon_t K_{i,t}^{\alpha} X_{i,t}^{1-\alpha} z_{t-1}^{\alpha}}{z_t z_{t-1}^{\alpha}} - \phi && (264)\\
&= \frac{z_t^{1-\alpha}\epsilon_t X_{i,t}^{1-\alpha} z_{t-1}^{\alpha}}{z_t}\frac{K_{i,t}^{\alpha}}{z_{t-1}^{\alpha}} - \phi && (265)\\
&= \frac{z_t^{1-\alpha}\epsilon_t X_{i,t}^{1-\alpha} z_{t-1}^{\alpha}}{z_t}\left(\frac{K_{i,t}}{z_{t-1}}\right)^{\alpha} - \phi && (266)\\
&= \frac{z_{t-1}^{\alpha}}{z_t^{\alpha}}\epsilon_t X_{i,t}^{1-\alpha} k_t^{\alpha} - \phi && (267)\\
&= \left(\frac{1}{\mu_{z,t}}\right)^{\alpha} \epsilon_t X_{i,t}^{1-\alpha} k_t^{\alpha} - \phi && (268)
\end{aligned}$$

in steady-state:

$$\begin{aligned}
y &= \left(\frac{1}{\mu_z}\right)^{\alpha} X^{1-\alpha} k^{\alpha} - \phi\\
&= \left(\frac{1}{\mu_z}\right)^{\alpha} X^{1-\alpha} k^{\alpha} - (\lambda_f - 1)y
\end{aligned}$$

Solve for y :

$$\begin{aligned}
y\left(1 + (\lambda_f - 1)\right) &= \left(\frac{1}{\mu_z}\right)^{\alpha} X^{1-\alpha} k^{\alpha}\\
y &= \frac{1}{\lambda_f}\left(\frac{1}{\mu_z}\right)^{\alpha} X^{1-\alpha} k^{\alpha}
\end{aligned}$$

:

Differentiate the production function w.r.t. all of the arguments:

$$\frac{\partial \left(\left(\frac{1}{\mu_{z,t}} \right)^{\alpha} \epsilon_t X_{i,t}^{1-\alpha} k_t^{\alpha} - \phi \right)}{\partial \mu_{z,t}}$$

$$\frac{\partial y_{it}}{\partial y_{it}} = 1$$

$$\begin{aligned} \frac{\partial y_{it}}{\partial \mu_{z,t}} &= -\mu_{z,t}^{-\alpha-1} \alpha \epsilon_t X_{i,t}^{1-\alpha} k_t^{\alpha} \\ ss &: -\mu_z^{-\alpha-1} \alpha X^{1-\alpha} k^{\alpha} \end{aligned}$$

$$\begin{aligned} \frac{\partial y_{it}}{\partial \epsilon_t} &= \left(\frac{1}{\mu_{z,t}} \right)^{\alpha} X_{i,t}^{1-\alpha} k_t^{\alpha} \\ ss &: \left(\frac{1}{\mu_z} \right)^{\alpha} X^{1-\alpha} k^{\alpha} \end{aligned}$$

$$\begin{aligned} \frac{\partial y_{it}}{\partial X_{i,t}} &= (1-\alpha) \left(\frac{1}{\mu_{z,t}} \right)^{\alpha} \epsilon_t X_{i,t}^{-\alpha} k_t^{\alpha} \\ ss &: (1-\alpha) \left(\frac{1}{\mu_z} \right)^{\alpha} X^{-\alpha} k^{\alpha} \end{aligned}$$

$$\begin{aligned} \frac{\partial y_{it}}{\partial k_{i,t}} &= \alpha \left(\frac{1}{\mu_{z,t}} \right)^{\alpha} \epsilon_t X_{i,t}^{1-\alpha} k_t^{\alpha-1} \\ ss &: \alpha \left(\frac{1}{\mu_z} \right)^{\alpha} X^{-\alpha} k^{\alpha-1} \end{aligned}$$

Collect terms:

$$y\hat{y}_t = -\mu_z^{-\alpha}\alpha X^{1-\alpha}k^{\alpha}\hat{\mu}_{z,t}+\left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}\hat{\epsilon}_t+(1-\alpha)\left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}X\hat{X}_t+\alpha\left(\frac{1}{\mu_z}\right)^{\alpha}X^{-\alpha}k^{\alpha-1}k\hat{k}_t$$

or

$$y\hat{y}_t = \left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}\left(-\alpha\hat{\mu}_{z,t}+\hat{\epsilon}_t+(1-\alpha)\hat{X}_t+\alpha\hat{k}_t\right)$$

$$y\hat{y}_t = \left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}\left(-\alpha\hat{\mu}_{z,t}+\hat{\epsilon}_t+(1-\alpha)\hat{X}_t+\alpha\hat{k}_t\right)$$

Make use of:

$$y = \frac{1}{\lambda_f}\left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}$$

$$\frac{1}{\lambda_f}\left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}\hat{y}_t = \left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}\left(-\alpha\hat{\mu}_{z,t}+\hat{\epsilon}_t+(1-\alpha)\hat{X}_t+\alpha\hat{k}_t\right)$$

Divide by $\left(\frac{1}{\mu_z}\right)^{\alpha}X^{1-\alpha}k^{\alpha}$

$$\frac{1}{\lambda_f}\hat{y}_t = \left(-\alpha\hat{\mu}_{z,t}+\hat{\epsilon}_t+(1-\alpha)\hat{X}_t+\alpha\hat{k}_t\right)$$

Multiply by λ_f:

$$\begin{aligned}\hat{y}_t &= \lambda_f\left((1-\alpha)\hat{X}_t+\alpha\left(\hat{k}_t-\hat{\mu}_{z,t}\right)+\hat{\epsilon}_t\right)\\ &= \lambda_f(1-\alpha)\hat{X}_t+\lambda_f\alpha\left(\hat{k}_t-\hat{\mu}_{z,t}\right)+\lambda_f\hat{\epsilon}_t\end{aligned}$$