Note on notations: Subscripts H and F denote whether the commodity originates in the Home or Foreign economy, respective. A '*' as a superscript indicates that the variable represents a value in the Foreign economy. E.g. $M_{H,t}^{*}(j^{*})$ denotes the nominal holdings of home currency by the foreign Household j^{*}.

Model Setup:

Home house lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t(j)^{1-\rho}}{1-\rho} + \chi \ln \frac{M_{H,t}(j)}{P_t} + V(G_t) - \frac{\kappa}{2} l_t(j)^2\right]$$

s.t. $\quad B_{t+1}(j) + M_{H,t}(j) = (1+i_t)B_t(j) + M_{H,t-1}(j) + W_t(j)l_t(j) - P_t T_t(j) - P_t C_t(j)$

Foreign household lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{*}(j^{*})^{1-\rho}}{1-\rho} + \chi^{*} \ln \frac{M_{F,t}^{*}(j^{*})}{P_t^{*}} + \eta^{*} \ln \frac{M_{H,t}^{*}(j^{*})}{P_t^{*} S_t} + V(G_t^{*}) - \frac{\kappa^{*}}{2} l_t^{*}(j^{*})^2\right]$$

s.t. $\quad \frac{B_{t+1}^{*}(j^{*})}{S_t} + M_{F,t}^{*}(j^{*}) + \frac{M_{H,t}(j^{*})}{S_t} = (1+i_t)\frac{B_t^{*}(j^{*})}{S_t} + M_{F,t-1}^{*}(j^{*}) + \frac{M_{H,t-1}(j^{*})}{S_t} +$
$W_t^{*}(j^{*})\, l_t^{*}(j^{*}) - P_t^{*} T_t^{*}(j^{*}) - P_t^{*} C_t^{*}\,(j^{*})$

Cost of consumption (composite) good (similar to its foreign analog):

$$\min_{C_{H,t}, C_{F,t}} P_t C_t = \min_{C_{H,t}, C_{F,t}} \{P_{H,t} C_{H,t} + P_{F,t} C_{F,t}\}$$

s.t. $\quad C_t \equiv (C_{H,t}(j))^{\gamma} \left(C_{F,t}(j)\right)^{1-\gamma}$

Firm's maximization problem (similar to its foreign analog):

$$\max_{l_t} \Pi_t = \max_{l_t} \left\{P_{H,t} Y_t - \int_0^1 W_t(j) l_t(j) dj\right\}$$

s.t. $\quad Y_t = \left(\int_0^1 l_t(j)^{\frac{\phi-1}{\phi}} dj\right)^{\frac{\phi}{\phi-1}}$

The solution for the firm's problem gives the following labor demand (similar to its foreign analog):

$$l_t(j) = \left(\frac{W_t(i)}{P_{H,t}}\right)^{-\phi} Y_t$$

Home governments budget constraint:

$$M_t - M_{t-1} + P_{H,t}\int_0^1 T_t(j)dj = P_{H,t}G_t$$

Where $M_t = \int_0^1 M_{H,t}(j) + \int_0^1 M_{H,t}^*(j^*)$

Foreing goverment budget constraint:

$$\int_0^1 M_{F,t}^*(j^*)dj^* - \int_0^1 M_{F,t-1}^*(j^*)dj^* + P_{H,t}^*\int_0^1 T_t^*(j^*)dj^* = P_{H,t}^*G_t^*$$

International bond market:

$$\int_0^1 B_t(j)dj + \int_0^1 B_t^*(j)dj^* = 0$$

Home and foreign resource constraints:

$$Y_t = G_t + \int_0^1 C_{H,t}(j)dj + \int_0^1 C_{H,t}^*(j^*)dj^*$$

$$Y_t^* = G_t^* + \int_0^1 C_{F,t}(j)dj + \int_0^1 C_{F,t}^*(j^*)dj^*$$

Equilibrium conditions: In our setup, agents are identical within each economy, hence, we can drop the 'j' indexes and deal with per capita variables.

Euler equations:

$$C_t^{-\rho} = \beta E_t\left[C_{t+1}^{-\rho}(1 + r_{t+1})\right]$$

$$C_t^{*-\rho} = \beta E_t\left[C_{t+1}^{*-\rho}(1 + r_{t+1})\right]$$

Money demand equations:

$$\frac{M_{H.t}}{P_t} = \chi E_t\left[\frac{1+i_{t+1}}{i_{t+1}} C_t^{\rho}\right]$$

$$\frac{M_{F,t}^*}{P_t^*} = \chi^* \mathrm{E_t}\left[\frac{S_t(1+i_{t+1})}{S_t(1+i_{t+1}) - S_{t+1}} C_t^{*\rho}\right]$$

$$\frac{M_{H,t}^*}{P_t^*} = \eta^* \mathrm{E_t}\left[\frac{1+i_{t+1}}{i_{t+1}} C_t^{*\rho} S_t\right]$$

Cost and benefit of holding money:

$$\frac{\chi}{M_{H,t}} = \frac{C_t^{-\rho}}{P_t} - \beta E_t\left[\frac{C_{t+1}^{-\rho}}{P_{t+1}}\right]$$

$$\frac{\chi^*}{M_{F,t}^*} = \frac{C_t^{*-\rho}}{P_t^*} - \beta E_t\left[\frac{C_{t+1}^{*-\rho}}{P_{t+1}^*}\right]$$

$$\frac{\eta^*}{M_{H,t}^*} = \frac{C_t^{*-\rho}}{P_t^* S_t} - \beta E_t\left[\frac{C_{t+1}^{*-\rho}}{P_{t+1}^* S_{t+1}}\right]$$

Wage setting equations:

$$E_{t-1}\kappa Y_t = \frac{\phi-1}{\phi} P_{H,t} E_{t-1}\left[\frac{C_t^{-\rho}}{P_t}\right]$$

$$E_{t-1}\kappa^* Y_t^* = \frac{\phi^*-1}{\phi^*} P_{F,t}^* E_{t-1}\left[\frac{C_t^{*-\rho}}{P_t^*}\right]$$

Households budget constraints:

$$B_{t+1} + M_{H,t} = (1+i_t)B_t + M_{H,t-1} + W_t l_t - P_t T_t - P_t C_t$$

$$\frac{B_{t+1}^*}{S_t} + M_{F,t}^* + \frac{M_{H,t}}{S_t} = (1+i_t)\frac{B_t^*}{S_t} + M_{F,t-1}^* + \frac{M_{H,t-1}}{S_t} + W_t^* l_t^* - P_t^* T_t^* - P_t^* C_t^*$$

Governments budget constraints:

$$M_t - M_{t-1} + P_t T_t = P_{H,t} G_t, \qquad Where\ \ M_t = M_{H,t} + M_{H,t}^*$$

$$M_{F,t}^* - M_{F,t-1}^* + P_t^* T_t^* = P_{H,t}^* G_t^*$$

Resources constraints:

$$Y_t = G_t + C_{H,t} + C_{H,t}^*$$

$$Y_t^* = G_t^* + C_{F,t}^* + C_{F,t}$$

Bond market condition:

$$B_t + B_t^* = 0$$

Consumption of tradable goods:

$$C_{H,t} = \frac{\gamma P_t}{P_{H,t}} C_t$$

$$C_{H,t}^* = \frac{\gamma P_t^*}{P_{H,t}^*} C_t^*$$

$$C_{F,t} = \frac{(1-\gamma)P_t}{P_{F,t}} C_t$$

$$C_{F,t}^* = \frac{(1-\gamma)P_t^*}{P_{F,t}^*} C_t^*$$

Prices indexes:

$$P_t = \frac{1}{\gamma_W} P_{H,t}^{\gamma} P_{F,t}^{1-\gamma}$$

$$P_t^* = \frac{1}{\gamma_W} P_{H,t}^{*\gamma} P_{F,t}^{*1-\gamma}$$

Purchasing power parity:

$$P_{H,t} = P_{H,t}^* S_t \ \ and\ \ P_{F,t} = P_{F,t} S_t$$

Output and nominal wages:

$$Y_t = l_t \ \ and\ \ Y_t^* = l_t^*$$

$$P_{H,t} = W_t \;\; and \;\; P^*_{F,t} = W^*_t$$

Fisher equation:

$$(1 + i_{t+1}) = (1 + r_{t+1}) \frac{P_{t+1}}{P_t}$$