RBC with government and preference shocks

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Consider the representative HH problem:

$$
\begin{aligned}
\max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}} U &= E_0\left\{\sum_{t=0}^{\infty}\beta^t[\ln(c_t)+\eta_t\ln(1-n_t)]\right\} \qquad (1)\\
s.t. &: c_t+i_t+T_t\leq w_tn_t+r_tk_t\\
&: k_{t+1}=(1-\delta)k_t+i_t\\
&: \eta_{t+1}=(1-\rho_\eta)\tilde{\eta}+\rho_\eta\eta_t+\varepsilon_{\eta,t+1}:\varepsilon_{\eta,t}\sim N(0,\sigma_\eta^2)\\
&: c_t,k_{t+1}\geq 0\ \forall t\\
&: k_0,z_0,G_0,\eta_0 \text{ given}
\end{aligned}
$$

The representative firm's problem:

$$
\begin{aligned}
\max_{\{n_t,k_t\}_{t=0}^{\infty}} \pi_t &= p_t(y_t-r_tk_t-w_tn_t) \qquad (2)\\
s.t. &: y_t=e^{z_t}k_t^{\theta}n_t^{1-\theta}\\
&: z_{t+1}=\rho_z z_t+\varepsilon_{z,t+1}:\ \varepsilon_{z,t}\sim N(0,\sigma_z^2)
\end{aligned}
$$

where Government's purchases are driven by the following stochastic process:

$$
\begin{aligned}
G_t &= T_t && (3)\\
G_{t+1} &= (1-\rho_G)\tilde{G} + \rho_G G_t + \varepsilon_{G,t+1} : \varepsilon_{G,t} \sim N(0,\sigma_G^2)
\end{aligned}
$$

Note that I depart from the original formulation in that the mean value of the preference and government spending shocks will be denoted by $\tilde{\eta}$ and $\tilde{G}$ rather than $\bar{\eta}$ and $\bar{G}$ since I use bars to denote steady state values.

[1].

Recursively, the problem can be written as:

Let K represent the aggregate capital stock in the economy.

Let k represent the capital stock of the representative household.

Let c be the numeraire good, making $r(z,G,\eta,K)$ and $w(z,G,\eta,K)$ the economy's rental rices for capital and labour in the terms of the numeraire good.

In turn, the problem solved by the representative HH is:

$$
\begin{aligned}
V(z,G,\eta,K,k) &= \max_{c,n,k'} \{\ln c + \eta \ln(1-n) + \beta E V(z',G',\eta',K',k')\} && (4)\\
s.t. &: w(z,G,\eta,K)n + r(z,G,\eta,K)k \geq c + k' - (1-\delta)k + G && (5)\\
&: \eta' = (1-\rho_\eta)\tilde{\eta} + \rho_\eta \eta + \varepsilon'_\eta : \varepsilon'_\eta \sim N(0,\sigma_\eta^2) && (6)\\
&: z' = \rho_z z + \varepsilon'_z : \ \varepsilon'_z \sim N(0,\sigma_z^2) && (7)\\
&: G' = (1-\rho_G)\tilde{G} + \rho_G G + \varepsilon'_G : \varepsilon'_G \sim N(0,\sigma_G^2) && (8)\\
&: K' = G(z,G,\eta,K) && (9)\\
&: c, k' \geq 0 && (10)
\end{aligned}
$$

The firm's problem would be:

$$\max_{n,k} \pi = y - r(z,G,\eta,K)k - w(z,G,\eta,K)n \tag{11}$$
$$s.t. \quad : \quad y = e^z k^\theta n^{1-\theta}$$

Replacing the budget constraint into the value function yields:

$$V(z,G,\eta,K,k) = \max_{n,k'} \{\ln[w(z,G,\eta,K)n + r(z,G,\eta,K)k - k' + (1-\delta)k - G] + \eta \ln(1-n) + \beta E V(z',G',\eta',K')\} \tag{12}$$

FOCs for the representative agent:

$$[k'] : \frac{1}{c} = \beta E V_k(z',G',\eta',K',k') \tag{13}$$

$$[n] : w(z,G,\eta,K) = \frac{\eta c}{(1-n)} \tag{14}$$

Envelope condition:

$$V_k(z,G,\eta,K,k) = \frac{[r(z,G,\eta,K) + (1-\delta)]}{c} \tag{15}$$

Replacing (15) into (13) yields the Euler equation:

$$1 = \beta E \left(\frac{c}{c'}\right) [r(z,G,\eta,K) + (1-\delta)] \tag{16}$$

In turn, the FOCs for the representative firm:

$$[n] : w(z,G,\eta,K) = (1-\theta) e^z \left(\frac{k}{n}\right)^\theta \tag{17}$$

$$[k] : r(z, G, \eta, K) = \theta e^{z} \left(\frac{k}{n} \right)^{\theta - 1} \tag{18}$$

Imposing equilibrium conditions:

Replacing (17) into (14) and (18) into (16), together with aggregate consistency conditions ($n = N$, $c = C$ and $k = K$), the law of motion of the shocks and the budget constraint yields the system of equations that characterize the solution to this model:

$$1 = \beta E_t \left(\frac{c}{c'} \right) [\theta e^{z} \left(\frac{K'}{N'} \right)^{\theta - 1} + (1 - \delta)] \tag{19}$$

$$(1 - \theta) e^{z} \left(\frac{K}{N} \right)^{\theta} = \frac{\eta c}{(1 - N)} \tag{20}$$

$$Y = c + K' - (1 - \delta) K + G \tag{21}$$

$$z' = \rho_z z + \varepsilon_z' \tag{22}$$

$$G' = (1 - \rho_G) \tilde{G} + \rho_G G + \varepsilon_G' \tag{23}$$

$$\eta' = (1 - \rho_\eta) \tilde{\eta} + \rho_\eta \eta + \varepsilon_\eta' \tag{24}$$

$$Y = e^{z} K^{\theta} N^{1 - \theta} \tag{25}$$

Note that I do not replace (21) into (25) because y_t will be one of the observables used in the estimation and as such we need one more equation to have a squared system of equations.

Calculating steady state values:

At the steady state, we know that:[1]

$$\begin{aligned}
C &= C' = \bar{C} \\
K &= K' = \bar{K} \\
N &= N' = \bar{N} \\
z &= z' = \bar{z} = 0 \\
\eta &= \eta' = \bar{\eta} = \tilde{\eta} \\
G &= G' = \bar{G} = \tilde{G}
\end{aligned}$$

In turn, we can re-write the above system as:

$$1 = \beta[\theta \left(\frac{\bar{K}}{\bar{N}}\right)^{\theta-1} + (1-\delta)] \tag{26}$$

$$(1-\theta)\left(\frac{\bar{K}}{\bar{N}}\right)^{\theta} = \frac{\bar{\eta}\bar{C}}{(1-\bar{N})} \tag{27}$$

$$\bar{Y} = \bar{K}^{\theta}\bar{N}^{1-\theta} = \bar{C} + \delta\bar{K} + \bar{G} \tag{28}$$

Rearranging (26) yields:

$$\frac{\bar{K}}{\bar{N}} = \left[\frac{1}{\theta}\left(\frac{1}{\beta} - (1-\delta)\right)\right]^{\frac{1}{\theta-1}} = P \tag{29}$$

Given $\bar{N} = 0.3$, we can use this to pin down $\bar{K}$:

[1] Note that, given $\varepsilon'_z = \varepsilon'_\eta = \varepsilon'_G = 0$:
$\bar{z} = \rho_z \bar{z} \Rightarrow \bar{z}(1-\rho_z) = 0 \Rightarrow \bar{z} = 0$
$\bar{\eta} = (1-\rho_\eta)\tilde{\eta} + \rho_\eta \bar{\eta} \Rightarrow \bar{\eta} = \tilde{\eta}$
$\bar{G} = (1-\rho_G)\tilde{G} + \rho_\eta \bar{G} \Rightarrow \bar{G} = \tilde{G}$

$$\bar{K} = 0.3\bar{P} \tag{30}$$

From (28) we obtain an expression for $\bar{Y}$:

$$\bar{Y} = \bar{K}^{\theta}\bar{N}^{1-\theta} \tag{31}$$

Also, given $\left(\frac{\bar{G}}{\bar{Y}}\right) = 0.17$ we can pin down $\bar{G}$. Moreover, from (28) we can derive an expression for $\bar{C}$:

$$\bar{Y} = \bar{C} + \delta\bar{K} + \bar{G} \tag{32}$$

Last, from (28) we obtain an expression for the steady state value of η:

$$\bar{\eta} = \frac{(1-\theta)(1-\bar{N})P^{\theta}}{\bar{C}} \tag{33}$$

Finally, given the parameter values we can compute the steady states:

$$\begin{aligned}
\bar{K} &= 11.3968 \\
\bar{C} &= 0.6376 \\
\bar{N} &= 0.3 \\
\bar{\eta} &= 2.6035 \\
\bar{Y} &= 1.1112 \\
\bar{G} &= 0.1889
\end{aligned}$$

Linearizing the model:

Below the linearization of the system of equations (19) to (25):

The first order Taylor expansion of the Euler equation (19) is as follows:

$$xx \tag{34}$$

For the intertemporal optimality condition (20) we have that:

$$xx \tag{35}$$

The linearization of the resource constraint (21): is as follows:

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t + G_t \tag{36}$$

Law of motion of the technological shock (22):

$$z_{t+1} = \rho_z z_t + \varepsilon_{z,t+1} \tag{37}$$

Law of motion of government spending (23):

$$G_{t+1} = (1-\rho_G)\tilde{G} + \rho_G G_t + \varepsilon_{G,t+1} \tag{38}$$

Law of motion of the agent's preference shock (24):

$$\eta_{t+1} = (1-\rho_\eta)\tilde{\eta} + \rho_\eta \eta_t + \varepsilon_{\eta,t+1} \tag{39}$$

Last, the production function (25):

$$\hat{y}_t = z_t + \theta\hat{k}_t + (1-\theta)\hat{n}_t \tag{40}$$